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Discontinuous Numerical Perturbation Algorithm for Convective-Diffusion Equation

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Abstract: The discontinuous numerical perturbation algorithm (for simplicity, it is called DP algorithm) is presented and tested by using various convective-diffusion equations. The DP algorithm is constructed by splitting the second order central difference schemes of both convective and diffusion terms of the convective-diffusion equation into upstream and downstream parts, then the perturbation reconstruction polynomials of grid interval for upstream and downstream convective terms are determined by using higher-order equations deduced from original equation and eliminating the truncated errors of the modified differential equation. The important nature, i.e. the upwind dominance nature, which is the basis to ensure the DP cetral schemes are stable and essentially oscillation free, is firstly presented and verified. Various numerical tests show that the DP central schemes are efficient, robust, and more accurate than the original second order central scheme.

Keywords: Discontinuous Numerical Perturbation Algorithms, Computational Fluid Dynamics, Convective Diffusion Equation.

1 Introduction

A number of numerical methods have been developed for computational fluid dynamics (CFD). The second order central difference scheme (2-CDS) has the comprehensive advantages of the accuracy, efficiency, simplicity, and maintainability, hence it is regarded as a good scheme for some CFD applications[1], for example, it is used for the large eddy simulation in the pressure-based solver in FLUENT. However, the 2-CDS scheme generates spurious oscillation if the solution contains a large gradient or a discontinuity. How to overcome this drawback and improve its accuracy and robustness, without increasing any nodes and the complexity, is no doubt meaningful and practical. The discontinuous numerical perturbation algorithm provides a new approach to achieve this destination.

The numerical perturbation algorithm is to couple fluid dynamics effects with the discretized schemes of the convective diffusion equation[2, 3]. The main steps of constructing the algorithm are as follows: the first order upwind scheme, the second order central schemes of convective derivative are reconstructed as a power-series of grid interval; by using the convective-diffusion equation itself, a series of higher-order fluid mechanics relation are obtained; by eliminating truncated error terms in the modified differential equation of the reconstructed scheme, the coefficients in the power-series are determined and finally the numerical perturbation algorithms are obtained.

Due to its physical preserving idea, and larger stable range and better accuracy than the original scheme, the numerical perturbation algorithm was applied to reconstruct various schemes with three- and five-nodes[4, 5, 6], and formed the high order perturbation difference scheme[2] and perturbation finite volume scheme[3, 7, 8] systems. Recently, based on the second order central difference scheme, Gao[9] proposed a stable perturbation finite difference scheme (in this paper, it is called high resolution numerical perturbation scheme, or DP scheme) for the convective-diffusion equation. First, the the second order difference schemes for both convetive and diffusion terms are split into two parts with upstream and downstream nodes, respectively, then the numerical perturbation is applied to reconstruct the split schemes. Numerical examples show that the new scheme is oscillation free even on coarse grids, and its errors are greatly less than the second order central scheme. In this paper, the natures of the DP central schemes are analyzed, and various numerical cases are calculated to verify the high performance of the DP central schemes.

2 Discontinuous Numerical Perturbation Central Scheme in One-dimensional Case

A general convective diffusion equation can be written as,

$$u\frac{\partial\phi}{\partial x}=\mu\frac{\partial^2\phi}{\partial x^2} \tag{1}$$

where, u, ϕ and μ denote the velocity, the transfer variable and the diffusion coefficient, respectively. If the second order central difference schemes are used to discretize both convective and diffusion terms in (1), the discretized equation is written as,

$$\frac{u_i}{2\Delta x}\left(\phi_{i+1}-\phi_{i-1}\right)=\frac{\mu}{\Delta x^2}\left(\phi_{i+1}-2\phi_i+\phi_{i-1}\right) \tag{2}$$

As mentioned previously, the second order central difference scheme (2) generates oscillation if the solution contains a large gradient or a discontinuity. The discontinuous numerical perturbation algorithm aims at eliminating the non-physical numerical oscillation and obtaining higher accurate solution. For completeness, the numerical perturbation algorithm[9] is briefly introduced. First, Eq.(2) is rewritten as

$$\left[\frac{u_i}{2\Delta x}\left(\phi_i-\phi_{i-1}\right)+\frac{\mu}{\Delta x^2}\left(\phi_i-\phi_{i-1}\right)\right]+\left[\frac{u_i}{2\Delta x}\left(\phi_{i+1}-\phi_i\right)-\frac{\mu}{\Delta x^2}\left(\phi_{i+1}-\phi_i\right)\right]=0 \tag{3}$$

The discontinuous numerical perturbation scheme is written as

$$\left[\frac{u_iG_i^+}{2\Delta x}\left(\phi_i-\phi_{i-1}\right)+\frac{\mu}{\Delta x^2}\left(\phi_i-\phi_{i-1}\right)\right]+\left[\frac{u_iG_i^-}{2\Delta x}\left(\phi_{i+1}-\phi_i\right)-\frac{\mu}{\Delta x^2}\left(\phi_{i+1}-\phi_i\right)\right]=0 \tag{4}$$

The functions $G_i^{\pm}$ are the power series of grid interval Δx

$$G_i^{\pm}=1+\sum_{n=1}^{N}a_n^{\pm}\Delta x^n \tag{5}$$

By using the convective-diffusion equation (1) and frozen the coefficient of the convective term, we can get the approximate relations of higher order derivatives of the transfer variable ϕ as the following

$$\frac{\partial^n\phi}{\partial x^n}=\left(\frac{u_i}{\mu}\right)^{n-1}\frac{\partial\phi}{\partial x} \tag{6}$$

Combined the Taylor series expansion and the method of undetermined coefficients(here, notice that, the terms of $(\Delta x)^{-1}$ and $(\Delta x)^0$ are eliminated by the corrected Eq.(4) and the convective-diffusion equation (2), respectively. G_i^+ and G_i^- are depended on the first and the second parts of (4), respectively.), coefficients

$a_n^{\pm}$ are obtained as

$$
\begin{aligned}
a_{2n}^{\pm} &= 0, \quad n = 1, 2, \cdots \\
a_1^{+} &= -a_1^{-} = \frac{1}{3!}\frac{u_i}{\mu} \\
a_3^{+} &= -a_3^{-} = -\frac{1}{3 \times 5!}\left(\frac{u_i}{\mu}\right)^3 \\
a_5^{+} &= -a_5^{-} = \frac{1}{3 \times 7!}\left(\frac{u_i}{\mu}\right)^5 \\
a_7^{+} &= -a_7^{-} = -\frac{3}{5 \times 9!}\left(\frac{u_i}{\mu}\right)^7 \\
&\cdots
\end{aligned}
\tag{7}
$$

Hence, the $(2N+1)$-th order (since $a_{2N}^{\pm} = 0$) perturbation scheme is obtained by applying

$$
G^{\pm}(N) = 1 + \sum_{n=1}^{2N-1} a_n^{\pm} \Delta x^n \tag{8}
$$

in Eq.(4).

In order to analyze the DP algorithm Eqs.(4) and (8) conveniently, the perturbation function Eq.(8) can be written as following, for example,

$$
G^{+}(4) = 1 + \frac{1}{3!}R_{\Delta x} - \frac{1}{3 \times 5!}R_{\Delta x}^3 + \frac{1}{3 \times 7!}R_{\Delta x}^5 - \frac{3}{5 \times 9!}R_{\Delta x}^7 \tag{9}
$$

$$
G^{-}(4) = 1 - \frac{1}{3!}R_{\Delta x} + \frac{1}{3 \times 5!}R_{\Delta x}^3 - \frac{1}{3 \times 7!}R_{\Delta x}^5 + \frac{3}{5 \times 9!}R_{\Delta x}^7 \tag{10}
$$

where $R_{\Delta x} = \frac{u_i \Delta x}{\mu}$ is the grid Reynolds number.

The numerical perturbation scheme of (4), (9) and (10) has both mechanical and mathematical meanings:

(1) Eq. (4) shows the clear upwind characteristic of the convective flow by splitting the spatial discretization into upstream and downstream parts.

(2) The perturbation functions $G^{\pm}(N)$ of Eqs. (9) and (10) are the polynomials of the grid Reynolds number, hence, the properties of DP schemes can be studied by using the grid Reynolds number.

(3) The spatial grid interval, which is the intrinsic small parameter in numerical calculation, is used as the perturbation parameter to construct the numerical perturbation schemes.

(4) The accuracy of DP schemes is the discretization accuracy of the whole convective diffusion equation, while the accuracy of traditional difference schemes is usually the approximate accuracy of the first-order or second-order derivative in the equation.

By analysing, the properties of the perturbation function $G^{\pm}(N)$ are listed,

(1) $\frac{1}{2}[G^{+}(N) + G^{-}(N)] = 1$;

(2) if $u_i > 0$, then $G^{+}(1) > G^{-}(1)$, $\quad G^{+}(3) > G^{-}(3)$;
if $u_i < 0$, then $G^{-}(1) > G^{+}(1)$, $\quad G^{-}(3) > G^{+}(3)$;

(3) if $0 < R_{\Delta x} < Rc(u_i > 0)$, then $G^{+}(2) > G^{-}(2)$;
if $0 > R_{\Delta x} > -Rc(u_i < 0)$, then $G^{-}(2) > G^{+}(2)$;

$G^{\pm}(4)$ has the same property as $G^{\pm}(2)$, except the Rc is different. Rc can be regarded as a critical grid Reynolds number, which is defined as a positive minimum value to make $G^{+}(N) = G^{-}(N)$.

For example, $Rc = \sqrt{60}$ is solved for $G^{\pm}(2)$.

Fig.1 shows the distribution of the perturbation function $G^{\pm}(N)$ vs the grid Reynolds number $R_{\Delta x}$. The above properties indicate that the third- and seventh-order DP schemes are upwind dominant, the fifth- and ninth-order DP schemes are conditionally upwind dominant in a proper range of $R_{\Delta x}$. The property of upwind dominance is the essential nature of the DP schemes, hence the DP schemes are essentially oscillation free. If $G^{\pm}(N) \geq 0$ is required, the form of DP schemes looks like the weighted scheme of the first order upstream and downstream schemes.

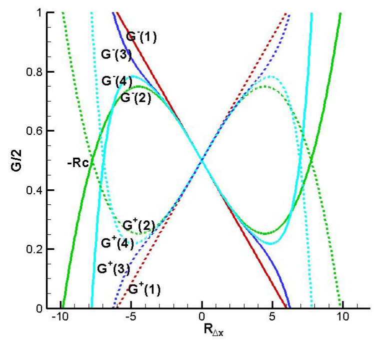

Figure 1: Perturbation function vs grid Reynolds number.

3 Discontinuous Numerical Perturbation Central Scheme in Two-dimensional Cases

Two-dimensional convective diffusion equation can be written as,

$$u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}=\mu\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) \tag{11}$$

The second order central difference schemes of the equation (11) is written as,

$$\begin{aligned}\frac{u_{ij}}{2\Delta x}\left(\phi_{i+1,j}-\phi_{i-1,j}\right)+\frac{v_{ij}}{2\Delta y}\left(\phi_{i,j+1}-\phi_{j,i-1}\right)=\\ \frac{\mu}{\Delta x^2}\left(\phi_{i+1,j}-2\phi_{ij}+\phi_{i-1,j}\right)+\frac{\mu}{\Delta y^2}\left(\phi_{j,i+1}-2\phi_{ij}+\phi_{j,i-1}\right)\end{aligned} \tag{12}$$

Scheme (12)'s splitting scheme in both the x- and y-space directions and the upstream-downstream splitting in the same sapce direction is

$$\begin{aligned}&\left(\frac{u_{ij}}{2\Delta x}+\frac{\mu}{\Delta x^2}\right)\left(\phi_{ij}-\phi_{i-1,j}\right)+\left(\frac{u_{ij}}{2\Delta x}-\frac{\mu}{\Delta x^2}\right)\left(\phi_{i+1,j}-\phi_{ij}\right)\\ +&\left(\frac{v_{ij}}{2\Delta y}+\frac{\mu}{\Delta y^2}\right)\left(\phi_{ij}-\phi_{i,j-1}\right)+\left(\frac{v_{ij}}{2\Delta y}-\frac{\mu}{\Delta y^2}\right)\left(\phi_{i,j+1}-\phi_{ij}\right)=0\end{aligned} \tag{13}$$

The discontinuous numerical perturbation scheme is written as

$$\begin{aligned}&\left(\frac{u_{ij}G_i^+}{2\Delta x}+\frac{\mu}{\Delta x^2}\right)\left(\phi_{ij}-\phi_{i-1,j}\right)+\left(\frac{u_{ij}G_i^-}{2\Delta x}-\frac{\mu}{\Delta x^2}\right)\left(\phi_{i+1,j}-\phi_{ij}\right)\\ +&\left(\frac{v_{ij}G_j^+}{2\Delta y}+\frac{\mu}{\Delta y^2}\right)\left(\phi_{ij}-\phi_{i,j-1}\right)+\left(\frac{v_{ij}G_j^-}{2\Delta y}-\frac{\mu}{\Delta y^2}\right)\left(\phi_{i,j+1}-\phi_{ij}\right)=0\end{aligned} \tag{14}$$

The functions $G_i^{\pm}$ are the power series of grid interval Δx and $G_j^{\pm}$ are the power series of grid interval Δy

$$G_i^{\pm}=1+\sum_{n=1}^{N}b_n^{\pm}\Delta x^n,\qquad G_j^{\pm}=1+\sum_{n=1}^{N}c_n^{\pm}\Delta y^n, \tag{15}$$

By using the space splitting equations of the convective-diffusion equation (11) and frozen the convective coefficient u_{ij} and v_{ij}, we can deduce the linear approximate relations of higher order derivatives of the transfer variable ϕ as the following

$$\frac{\partial^n \phi}{\partial x^n} = \left(\frac{u_{ij}}{\mu}\right)^{n-1} \frac{\partial \phi}{\partial x}, \qquad \frac{\partial^n \phi}{\partial y^n} = \left(\frac{v_{ij}}{\mu}\right)^{n-1} \frac{\partial \phi}{\partial y} \tag{16}$$

By using the higher-order approximate relations (16) and eliminating the truncated error terms of the modified differential equations of the DP scheme (14), we deduce all coefficients $b_n^{\pm}$ and $c_n^{\pm}$ as follows

$$\begin{aligned}
& b_{2n}^{\pm} = 0, \quad n = 1, 2, \cdots \\
& b_1^+ = -b_1^- = \frac{1}{3!}\frac{u_{ij}}{\mu} \\
& b_3^+ = -b_3^- = -\frac{1}{3\times 5!}\left(\frac{u_{ij}}{\mu}\right)^3 \\
& b_5^+ = -b_5^- = \frac{1}{3\times 7!}\left(\frac{u_{ij}}{\mu}\right)^5 \\
& b_7^+ = -b_7^- = -\frac{3}{5\times 9!}\left(\frac{u_{ij}}{\mu}\right)^7 \\
& \cdots \\
& c_{2n}^{\pm} = 0, \quad n = 1, 2, \cdots \\
& c_1^+ = -c_1^- = \frac{1}{3!}\frac{v_{ij}}{\mu} \\
& c_3^+ = -c_3^- = -\frac{1}{3\times 5!}\left(\frac{v_{ij}}{\mu}\right)^3 \\
& c_5^+ = -c_5^- = \frac{1}{3\times 7!}\left(\frac{v_{ij}}{\mu}\right)^5 \\
& c_7^+ = -c_7^- = -\frac{3}{5\times 9!}\left(\frac{v_{ij}}{\mu}\right)^7
\end{aligned} \tag{17}$$

Since $b_{2N}^{\pm} = c_{2N}^{\pm} = 0$, hence the $(2N+1)$-th order discontinuous perturbation scheme is obtained by letting

$$G_i^{\pm}(N) = 1 + \sum_{n=1}^{2N-1} b_n^{\pm} \Delta x^n, \qquad G_j^{\pm}(N) = 1 + \sum_{n=1}^{2N-1} c_n^{\pm} \Delta y^n \tag{18}$$

Obvously, the discontinuous perturbation central scheme of three-dimensionla convective diffusion equation can be deduced similarly.

4 Numerical Examples

Several cases are calculated to demonstrate the efficiency, robustness and high order accuracy of the DP schemes. In this paper, the time dependent method is applied to get the steady solution. The fourth order Runge-Kutta method is used for the time marching.

4.1 The Linear Convective Diffusion Equation

The linear convective diffusion equation is used as the first test case and can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1 \tag{19}$$

The comparison of the third order DP (3-DP) algorithm and the second order central difference scheme (2-CDS) with grid number $N = 160$ is shown in Fig. 2. Near the large gradient, 2-CDS causes dramatic oscillations. The 3-DP scheme is oscillation free and agree well with the exact solution. Table 1 shows the detailed error comparison of different schemes. In this table, OS denotes the oscillatory solution, OV denotes overflow happened in the calculation. It can be seen that, if a large grid interval (means large grid Reynolds

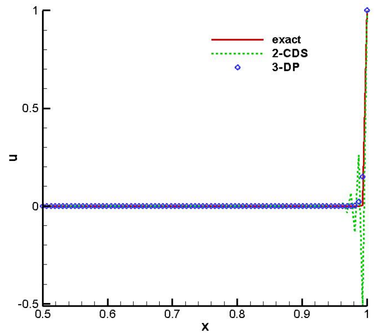

Figure 2: Linear convective diffusion equation, Re=1000.

number) is used, the second order central scheme is oscillatory. If $N = 80$ is used, the $R_{\Delta x}$ equals to 12.5, which is larger than R_c of $G(2)$ and $G(4)$, hence the 5-DP and 9-DP schemes are overflow. If $N = 160$ is used, the $R_{\Delta x}$ equals to 6.25, which is close to R_c, the 5-DP and 9-DP schemes are weak upwind dominant and result in a small oscillation near the large gradient region. Meanwhile, with the large grid Reynolds number(for example,$N = 80$), the 7-DP scheme is not necessarily more accurate than the 3-DP scheme. With the grid number increased, the DP schemes achieve the expected order (it is easy to calculate the accuracy order from the errors), for example, the third order, fifth order, seventh order, and ninth order, respectively.

4.2 The Viscous Burgers Equation

The viscous Burgers equation is the second test case. It is written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}, \quad a \leq x \leq b \tag{20}$$

$$u(a) = \tanh(-aRe/2), \quad u(b) = \tanh(-bRe/2) \tag{21}$$

The steady state solution of Eq. (20) with boundary condition (21) is $u(x) = \tanh(-xRe/2)$. At $x = 0$, the shock is formed with a large Re number. In our computation, $a = -1$, $b = 1$ are used. Based on the analysis of the 5th-order and 9th-order DP schemes are conditionally upwind dominant and the numerical results of previous cases, only the 3rd-order and the 7th-order schemes are considered for this case.

First, the case with $Re = 10$ is tested. Table 2 gives the errors. Since the small Reynolds number is used, all tested schemes obtain good results. It can also be seen that, the errors of the DP schemes are no more than one half of the second central scheme. Then, the case with $Re = 1000$ is calculated, and the errors are given in Table 3. The second order central scheme is oscillatory even with the finest grid number of $N = 640$. The DP schemes are oscillation free. Fig. 3 gives the comparison of the second order central scheme and the third order DP scheme with $Re = 1000$ and $N = 160$.

It is worth mentioning that, for this nonlinear case, the relation of high order derivatives (6) is approximate, hence the results of the high order(7th-order) DP scheme are almost the same as the low order (3rd-order) DP scheme. Hence, for the other cases in this paper, only the 2-CDS scheme and the 3-DP scheme are compared.

Table 1: The linear convective diffusion equation, $Re = 1000$

Scheme	N	L_∞ error	L_∞ order	L_1 error	L_1 order
2-CDS	80	OS			
	160	OS			
	320	OS			
	640	0.8629e-1		0.1948e-3	
3-DP	80	0.3743e+0		0.7386e-2	
	160	0.1479e+0	1.34	0.1083e-2	2.77
	320	0.2996e-1	2.30	0.1056e-3	3.36
	640	0.3080e-2	3.28	0.7742e-5	3.77
5-DP	80	OV			
	160	OS			
	320	0.7078e-2		0.2399e-4	
	640	0.1782e-3	5.31	0.4460e-6	5.75
7-DP	80	0.8750e+0		0.8639e-1	
	160	0.1254e+0	2.80	0.8945e-3	6.59
	320	0.1687e-2	6.22	0.5771e-5	7.28
	640	0.1080e-4	7.29	0.2704e-7	7.74
9-DP	80	OV			
	160	OS			
	320	0.4116e-3		0.1405e-5	
	640	0.6611e-6	9.28	0.1656e-8	9.73

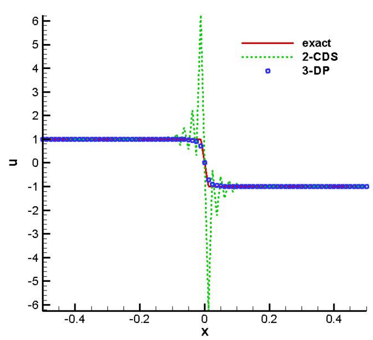

Figure 3: Nonlinear Burgers equation, Re=1000.

4.3 The Variable Coefficients Convective Diffusion Equation

The variable coefficients convective diffusion equation,

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}, \quad -1 \le x \le 1 \tag{22}$$

with the boundary condition

$$u(-1) = 1, \quad u(1) = -1 \tag{23}$$

is the third test case. The comparison of the steady state solution with $Re = 1000$ and $N = 160$ is given in Fig. 4. Where, the exact solution is the result calculated by the third order DP scheme with $N = 1000$. The second central scheme is still oscillatory, while the third order DP scheme obtains a good solution.

Table 2: The viscous Burgres equation, $Re = 10$

Scheme	N	L_∞ error	L_∞ order	L_1 error	L_1 order
2-CDS	80	0.2223e-2		0.6919e-3	
	160	0.5667e-3	1.97	0.1756e-3	1.98
	320	0.1427e-3	1.99	0.4404e-4	2.00
	640	0.3517e-4	2.02	0.1080e-4	2.03
3-DP	80	0.9514e-3		0.2498e-3	
	160	0.2434e-3	1.97	0.6341e-4	1.98
	320	0.6093e-4	2.00	0.1575e-4	2.01
	640	0.1463e-4	2.06	0.3696e-5	2.09
7-DP	80	0.9522e-3		0.2501e-3	
	160	0.2434e-3	1.97	0.6343e-4	1.98
	320	0.6093e-4	2.00	0.1575e-4	2.01
	640	0.1463e-4	2.06	0.3696e-5	2.09

Table 3: The viscous Burgres equation, $Re = 1000$

N	2-CDS		3-DP		7-DP	
	L_∞	L_1	L_∞	L_1	L_∞	L_1
80	0.1143e+2	0.5682e+0	0.4688e+0	0.2720e-1	0.7099e+0	0.2713e+0
160	0.5250e+1	0.1133e+0	0.2848e+0	0.5511e-2	0.4374e+0	0.2872e-1
320	0.2129e+1	0.1962e-1	0.1200e+0	0.8816e-3	0.8439e-1	0.6077e-3
640	0.6467e+0	0.2378e-2	0.4734e-2	0.2220e-4	0.3154e-1	0.1031e-3

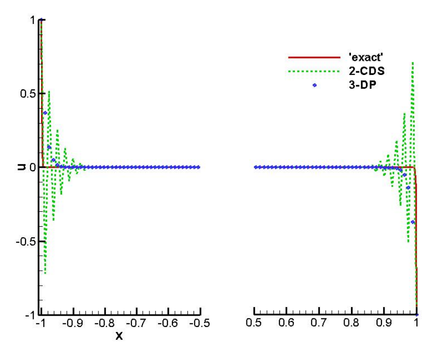

Figure 4: Variable coefficients convective diffusion equation, Re=1000.

4.4 The Convective Diffusion Equation with Source Terms

The convective diffusion equation with source terms is the fourth test case,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \epsilon\pi^2 \sin \pi x + \pi \cos \pi x, \quad 0 \le x \le 1 \tag{24}$$

The exact solution is given by $u(x) = \sin \pi x + (e^{x/\epsilon} - 1)/(e^{1/\epsilon} - 1)$. Fig. 5 plots the results of $Re = 1000$ and $N = 320$. This case shows that the DP scheme can also solve the convective diffusion equation with source terms very well.

Table 4: 2-D nonlinear Burgers equation, $Re = 1000$

$N_x \times N_y$	2-CDS		3-DP	
	L_∞	L_1	L_∞	L_1
40×40	OV		0.6420e+0	0.1434e+0
80×80	OV		0.4688e+0	0.2597e-1
160×160	OV		0.2848e+0	0.5440e-2
320×320	OV		0.1213e+0	0.8887e-3

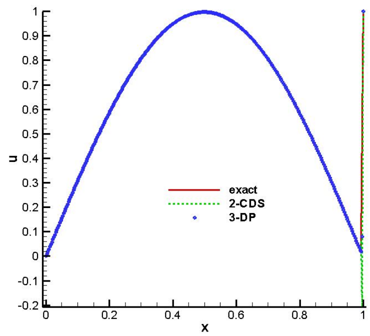

Figure 5: Convective diffusion equation with source terms, Re=1000.

4.5 The Two Dimensional Nonlinear Burgers Equation

The two dimensional nonlinear Burgers equation is the fifth test case,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad -1 \leq x \leq 1 \tag{25}$$

The exact solution is given by $u(x, y) = \tanh(-Re(x + y)/2)$. The second order central scheme is overflow even the grid of 320×320 is used. Table 4 gives the errors of the 3-DP scheme. Fig. 6 is the solution obtained by the 3-DP scheme with the grid of 160×160.

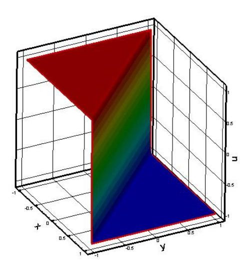

Figure 6: 2-D nonlinear Burgers equation, Re=1000, 3-DP scheme.

Table 5: 2-D convective diffusion equation with variable coefficients, $Re = 40$

$N_x \times N_y$	2-CDS		3-DP	
	L_∞	L_1	L_∞	L_1
40×40	0.3740e+2	0.1028e+1	0.1199e+0	0.2856e-2
80×80	0.1077e+2	0.2758e+0	0.8903e-2	0.2022e-3
160×160	0.2774e+1	0.7072e-1	0.5997e-3	0.1354e-4

4.6 The Two Dimensional Linear Convective Diffusion Equation with Variable Coefficients

Finally, the two dimensional linear convective diffusion equation with variable coefficients is calculated,

$$\frac{\partial u}{\partial t} + \left(y - \frac{1}{2}\right)\frac{\partial u}{\partial x} + \left(x - \frac{1}{2}\right)\frac{\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 \le x, y \le 1 \tag{26}$$

The exact solution is given by $u(x, y) = exp\left[Re(x - \frac{1}{2})(y - \frac{1}{2})\right]$. The case with $Re = 40$ is calculated. Since the grid Reynolds number is small relatively, the 2-CDS scheme and the 3-DP scheme can run this case. However, the errors given in Table 5 show that the second central scheme gives unacceptable results even with the mesh of 160×160, while the third order DP scheme obtains very good solution and it achieves third order accuracy. Fig. 7 is the solution calculated by the 3-DP scheme with mesh of 80×80.

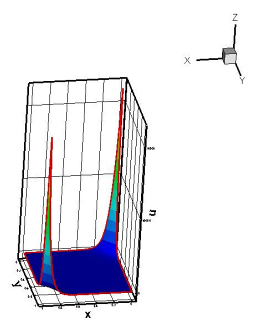

Figure 7: 2-D convective diffusion equation with variable coefficients, Re=40, 3-DP scheme.

5 Conclusions

The nature of upwind dominance or conditionally upwind dominance of the DP schemes is presented and verified in this paper. It is this nature that makes the DP schemes be essentially non-oscillatory schemes in the large gradient region or near discontinuity even with a large grid Reynolds number. Various numerical examples show that the DP schemes are not only efficient and robust, but also more accurate than the original second order central scheme. The application to fluid dynamics Navier-Stokes equations and the multi-nodes DP reconstruction algorithm are currently underway and will be reported at an upcoming paper.

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