The multi-dimensional limiters for discontinuous Galerkin methods on unstructured grids

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Abstract: High order limiters remain one of the main challenges for discontinuous Galerkin (DG) methods in solving hyperbolic conservation laws. This paper proposes an efficient limiting procedure for the DG method. The key feature is to construct additional polynomials from the solutions on neighboring cells by means of secondary reconstruction. Then the limited solution on current cell can be obtained using WENO or other limiting procedures. This limiting procedure uses only the face-neighbor information and thus is compact and easy to generalize to multi-dimensions. The numerical experiments show that the limiter can achieve high order accuracy in smooth region and also capture the strong discontinuities without oscillations.

Keywords: Discontinuous Galerkin method, limiters, unstructured grids.

1 The limiting procedure

For the hyperbolic conservation laws, $\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0$, We approximate the solution $U(\vec{x})$ with $U^h(\vec{x})$, which is composed by a polynomial of order k defined on Ω, i.e.,$U^h(\vec{x})=\sum_{l=0}^{K-1}U_i^l\varphi_{l,i}$, where K is the number of degree of freedom (DOF) and φ is a set of orthogonal basis. The DG discretization gives a series of equations to determine the coefficients U_i^l,

$$\frac{\partial U_i^l}{\partial t}\int\limits_{\Omega_i}(\varphi_{l,i})^2 d\Omega+\int\limits_{\partial\Omega_i}\Big(F(U^h),G(U^h)\Big)\cdot\vec{n}_e\varphi_{l,i}ds-\int\limits_{\Omega_i}\Big(F(U^h),G(U^h)\Big)\cdot\nabla\varphi_{l,i}d\Omega=0. \quad (1)$$

Since the numerical solution U_h is discontinuous across the faces, the numerical flux is computed in terms of Riemann solvers, such as HLL flux which is used in this paper.

To stabilize the DG scheme, methods should be introduced to resolve the discontinuities without spurious oscillations. In this paper, a novel limiting procedure is developed, which consists of the following two steps: (1) The addition candidate polynomials are obtained through the so-called secondary reconstruction in terms of the face-neighbors of current cell. The secondary reconstruction is introduced in [1] in the framework of the finite volume method, and is extended to the DG scheme in the present paper. (2) The WENO procedure is adopted to compute the weighted average of the unlimited solution of current cell and polynomials of the secondary reconstructions.

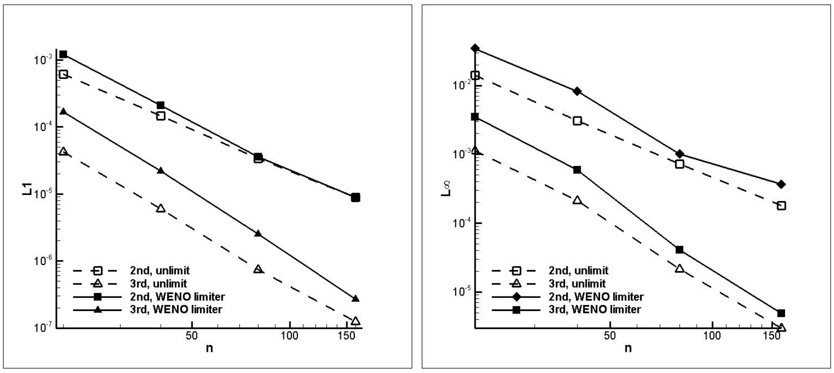

Figure 1: Accuracy for isentropic vortex problem for DG method with/without limiter.

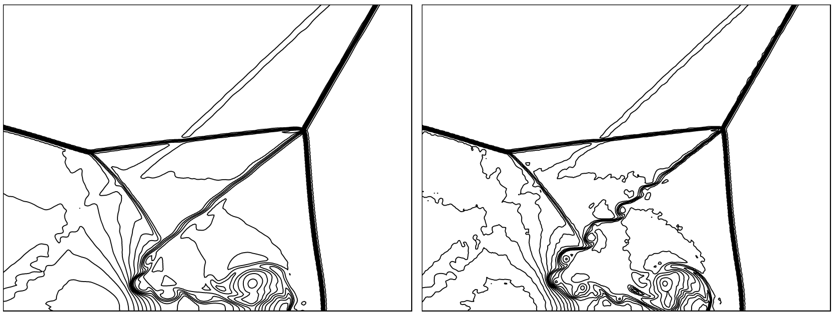

Figure 2: Mach stem with grid size $h = 1/240$. Left: 2nd order;right: 3rd order.

The secondary reconstruction is very simple and straightforward. Assuming cell j is a face-neighbor of cell i, we define the secondary reconstruction of cell i using the information of cell j as $u_{i,j}(\vec{x})$. $u_{i,j}(\vec{x})$ is constructed by solving

$$\begin{aligned} &\int_{\Omega_i} u_{i,j}(\vec{x})d\Omega = u_i^0 \left\|\Omega_i\right\|, \\ &\int_{\Omega_j} u_{i,j}(\vec{x})\varphi_{l,j}d\Omega = \int_{\Omega_j} u_j(\vec{x})\varphi_{l,j}d\Omega = u_j^k \int_{\Omega_j} (\varphi_{l,j})^2 d\Omega,\ l = 1, \ldots, K-1. \end{aligned} \tag{2}$$

where $\|\Omega_i\|$ is the area in Ω_i and u_i^0 is the cell average of the solution on cell i. It is clear that the secondary reconstruction is conservative and uses only information on current cell and its face-neighbors. Also the secondary reconstruction can be derived analytically and is therefore very efficient.

2 Numerical examples

Several numerical test cases have been solved to verify the accuracy preserving property and the performance of shock capturing. Fig. 1 shows the errors of the isotropic vortex problem and Fig. 2 is the results of the double Mach reflection problem.

References

[1] W. Li, Y-X Ren. High order k-exact WENO finite volume schemes for solving gas dynamic Euler equations on unstructured grids.*Int. J. Numer. Meth. Fluids.*, doi: 10.1002/fld.2710