# Very-High-Order Conservative Discretization of Diffusive Terms with Variable Viscosity

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**Abstract:** We study very-high-order conservative discretizations for diffusive terms with variable viscosity, which are present in the compressible Navier-Stokes equations, using viscous fluxes at cell-interfaces. We show that the proposed approach yields $O(\Delta x^{2s})$ accuracy on the stencil $\{i-s,\cdots,i,\cdots,i+s\}$, thus improving upon previous proposals which are $O(\Delta x^{2\lceil\frac{s}{2}\rceil})$ on the same stencil. The extension of the method to 2-D and 3-D regular cartesian grids is described. Several typical 1-D and 2-D computational examples substantiate the accuracy of the method for test problems and for DNS of turbulent flows using the 3-D compressible Navier-Stokes equations.

*Keywords:* High-Order Schemes, Viscous Terms, Diffusion Equation, Compressible Navier-Stokes.

# 1 Introduction

Very-high-order accuracy is essential in several practical applications such as DNS of compressible turbulence [3]. Whereas several very-high-order approaches for the discretization of convective terms have been developed [2], the conservative discretization of the diffusive (viscous) terms has received less attention. The popular compact scheme of Lele [4] is nonconservative. Zingg et al. [6] have developed a conservative scheme for the viscous terms on the stencil $\mathrm{s}_{i,3,3} := \{i-3,\cdots,i+3\}$, which yields an $O(\Delta x^4)$-accurate approximation of $\big(\mu(x)\ u'(x)\big)'$, and Shen et al. [5] developed an alternative $O(\Delta x^4)$-accurate conservative formulation on the same stencil. It is straightforward to generalize these apporaches to higher-order using larger stencils, obtaining $O(\Delta x^{2\lceil\frac{s}{2}\rceil})$ schemes on the stencil $\mathrm{s}_{i,s,s} := \{i-s,\cdots,i+s\}$. In the present work we develop an $O(\Delta x^{2s})$ method on the stencil $\mathrm{s}_{i,s,s} := \{i-s,\cdots,i+s\}$, *ie* twice more accurate.

# 2 Present Approach

To discretize $(\mu(x)u'(x))'_i$ on a homogeneous grid $x := x-1+(i-1)\Delta x$ we define the numerical flux $\check{F}_{(\mu u';i,s-1,s)_{i+\frac{1}{2}}}$ on the stencil $\mathrm{s}_{i,s-1,s} := \{i-s+1,\cdots,i+s\}$ satisfying

$$(\mu(x)u'(x))'_i = \frac{1}{\Delta x}\Big(\check{F}_{(\mu u';i,s-1,s)_{i+\frac{1}{2}}} - \check{F}_{(\mu u';i,s,s-1)_{i-\frac{1}{2}}}\Big) + O(\Delta x^{2s}) \tag{1a}$$

Let $p_{I,M_-,M_+}(x;x_i,\Delta x;f)$ be the Lagrange interpolating polynomial of $f:\mathbb{R}\longrightarrow\mathbb{R}$ on the stencil $\mathrm{s}_{i,M_-,M_+} := \{i-M_-,\cdots,i+M_+\}$ and $p_{R_1,M_-,M_+}(x;x_i,\Delta x;f)$ the corresponding reconstructing polynomial [1] which approximates the function $h:\mathbb{R}\longrightarrow\mathbb{R}$, whose cell-averages are equal to $f(x)$ ($f(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} h(x+\zeta\Delta x)d\zeta\ \forall x$). Then we can show analytically and verify computationally (Fig. 1) that the required numerical flux is

$$\check{F}_{(\mu u';i,s-1,s)_{i+\frac{1}{2}}} := p_{R_1,s-1,s}\Big(x_i+\tfrac{1}{2}\Delta x;x_i,\Delta x;\big[p_{I,s-1,s}(x;x_i,\Delta x;\mu)p'_{I,s-1,s}(x;x_i,\Delta x;f)\big]\Big) \tag{1b}$$

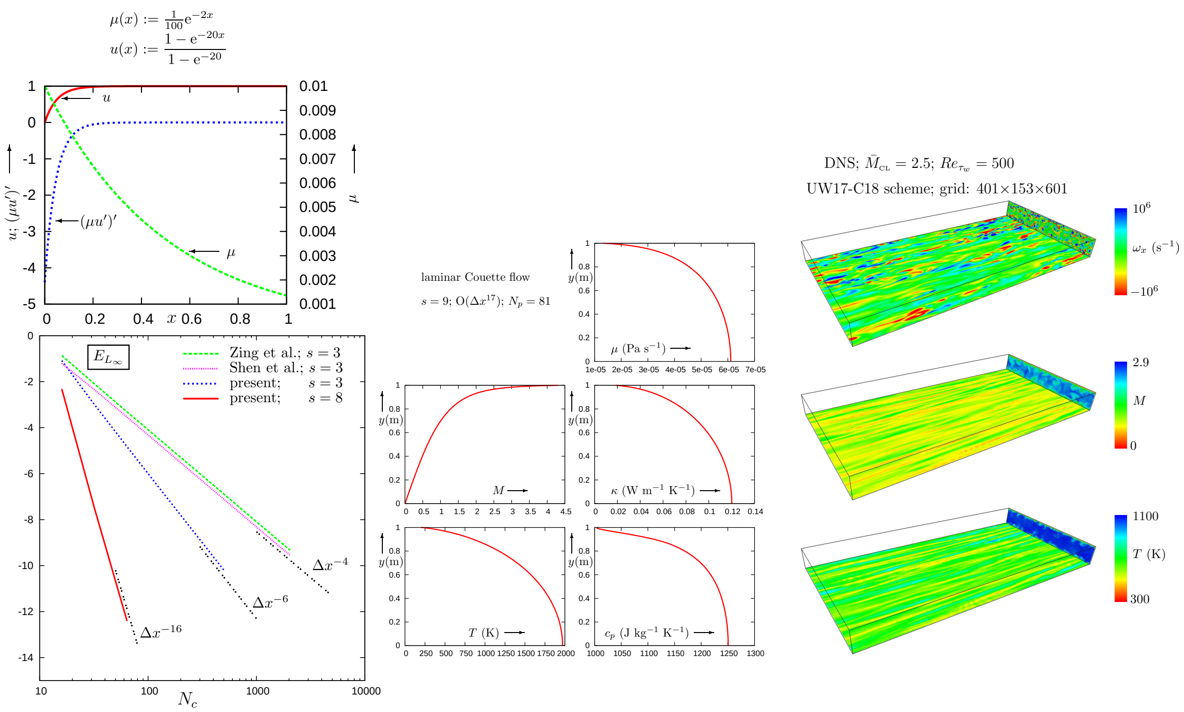


Figure 1: (Left) Error of the present approximation for the computation of $(\mu u')'$ as a function of grid refinement and comparison with previous approaches [5, 6]. (Center) High-Mach-number Couette flow testcase. (Right) Application to DNS of compressible channel flow.

$$\check{F}_{(\mu u';i,s-1,s)_{i+\frac{1}{2}}} = \frac{1}{\Delta x} \sum_{p=-s+1}^{s} \mu_{i+p} \sum_{q=-s+1}^{s} \left( \sum_{\ell=-s+1}^{s} \alpha_{R_1,s-1,s,\ell}(\tfrac{1}{2}) \alpha_{I,s-1,s,p}(\ell) \alpha'_{I,s-1,s,q}(\ell) \right) u_{i+q} \qquad (1c)$$

The improvement upon previous approaches [5, 6] comes from the fact that we do not reconstruct fluxes from interpolatory approximations of the product $\mu u'$ at half-points, but instead at the integer points of the stencil. At boundary-points, we use biased stencils recovering global $O(\Delta x^{2s-1})$ accuracy. The method is extended to 2-D and 3-D using the usual linewise approach [3, 5, 6]. Typical applications presented in the complete paper include:

1) Nonisothermal flow of glycerol (whose viscosity varies exponentially with temperature $T$)
2) Compressible laminar Couette flow (Fig. 1)
3) 2-D diffusion equation
4) DNS computations (Fig. 1).

# 3 Conclusion and Future Work

The present work defines numerical fluxes for very-high-order conservative discretization of $(\mu u')$, applicable to the viscous terms of the Navier-Stokes equations. Future work includes a least-squares genuinely multidimensional approach applicable to arbitrary unstructured grids and the development of WENO discretizations of these terms for flows with discontinuities.

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