# Rivulet evolution in gravity-driven thin-film flows

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**Abstract:** The problem of rivulet formation in the case of thin film flow down an inclined rigid substrate is investigated numerically and comparisons drawn with flow visualisations obtained experimentally. New results are presented for the evolving three-dimensional flow; the critical wavelength for the onset of instability, as a function of the inclination angle, is extracted and compared with predictions of the same based on a linear stability analysis.

*Keywords:* Multigrid, spatio-temporal adaptation, rivulets, lubrication theory.

## 1 Introduction

The motion of a thin film down an inclined planar substrate can result in complex behaviour and interesting dynamics at the associated advancing front which becomes unstable, forming a periodic pattern of finger shaped rivulets. Huppert [1] was the first to make a detailed study of the problem, showing the critical wavelength of the emerging instability, when scaled with the capillary length of the fluids considered, to be captured by a single linear fit of the data. Subsequent theoretical investigations have concentrated, in the main, on a linear stability analysis of the travelling wave solution; this approach does, however, prove inadequate [2] for substrates with a low inclination angle. An alternative approach is to explore the full three-dimensional problem, requiring a numerical solution of the associated governing equations. This is the route adopted below, based on the simplifying assumption that the flow can considered lubrication like [3].

## 2 Problem Formulation and Method of Solution

The problem of interest consists of a thin fluid film, of asymptotic thickness $H_0$, flowing down a substrate inclined at an angle $\theta$ to the horizontal, with a volumetric flow rate $Q_0$ per unit width. The fluid is considered to be incompressible with constant density, $\rho$, viscosity, $\mu$, and surface tension, $\sigma$, and the film to be fully wetting. Under the assumption that $H_0$ is small compared to the capillary length, $L_0 = H_0/(6Ca)^{\frac{1}{3}}$, where $Ca$ $O(\epsilon^3) << 1$ is the capillary number ($= \mu U_0/\sigma$), that is $H_0/L_0 << 1$, taking $U_0 = 3Q_0/2H_0$ and adopting appropriate scaling, [3], the governing Navier-Stokes and continuity equations reduce to a coupled system of partial differential equations for the film thickness, $h$, and pressure, $p$. For the case of fully developed flow upstream where the film thickness $H_0$ is fixed, the remaining boundary conditions which

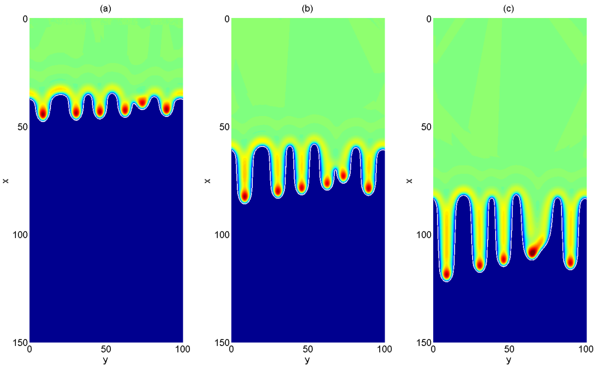

Figure 1: Colour maps of free surface height for the evolving gravity-driven flow of a water film, $H_0 = 100\mu$m ($\rho = 1000$ kg m$^{-3}$, $\mu = 0.001$ Pa s, $\sigma = 0.07$ N m$^{-1}$), over a planar substrate inclined at 65°: (a) t = 50, (b) t = 100 and (c) t = 150.

close the problem are as prescribed in [5]; initially, at time zero, the front itself is perturbed with a superposition of $N$ modes with random length, $l_j$, and differing wavelength, $\lambda_{0,j}$, as in [4] via $h(x,y) = 0.5\left\{1 + h^* - (1 - h^*)\tanh\left[x - x_f(y)\right]\right\}$; $x_f\left(y\right) = x_u - \sum_{j=1}^{N} l_j \cos\left(2\pi y/\lambda_{0,j}\right)$, where $x_u$ is the position of the unperturbed front.

Given the extent of the solution domain involved and the long-time solutions required, a key feature of the methodolgy used to solve the discretised governing equations, is one based on a strategy employing automatic error controlled adaptive time stepping and mesh refinement/derefinement within an efficient multigrid framework; further details are available in [5]. Noting that sufficiently far away from the advancing front, the film thickness remains constant, provides another avenue for exploitation, in that judiciously removing nodes in such regions has a dramatic effect in terms of further reducing the solution time without loss of accuracy.

## 3 Results

The predicted rivulet formation for long times is found to be in very good agreement with corresponding experiments. The shape and dynamics of the rivulet pattern change with inclination angle; of particular note is the merger of neighbouring rivulets in some instances; from the resulting numerical data an expression for the critical wavelength for the on-set of merging is obtained, in terms of the inclination angle (up to and including a vertically aligned substrate). Figure 1 provides an typical example of the evolutionary formation of rivulets at the advancing front of a thin water film; the merger of two neighbouring rivulets is clearly observable.

## References

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