A simple two-step Riemann solver that separates acoustic waves from contact and shear waves

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Abstract: The Euler equations consist of two acoustic waves ($u \pm c$), contact and shear waves moving at the speed of (u). The acoustic waves are an essential ingredient in compressible flows, but trivial in low Mach number flows. However, most of the upwind schemes treats these waves in the same fashion. In this paper, we propose a method to separate the acoustic waves from the advection waves, resulting in a simple and unique Riemann solver. It is of great advantage for the two-step method to allow different solution-strategies for each step.

Keywords: approximate Riemann solver, upwind scheme, Lagrange-Remap, compressible flows

Consider the one-dimensional system of conservation laws for any fluids,

$$\mathbf{U}_t + \mathbf{F}_x = 0, \tag{1}$$

where $\mathbf{U}$, $\mathbf{F}$ are vectors of conservative quantities and fluxes. The flux vector can be written as

$$\mathbf{F} = u\mathbf{U} + \mathbf{P}, \tag{2}$$

where $\mathbf{P} = (0, p, pu)^T$. For the numerical solution of (1), we shall consider a conservative scheme

$$\Omega_i \mathbf{U}_i^{n+1} = \Omega_i \mathbf{U}_i^n - \Delta t(\mathbf{F}^*_{i+1/2} - \mathbf{F}^*_{i-1/2}), \tag{3}$$

where Δt and Ω_i is the time step and the cell volume respectively. Conservative schemes are different at the way to define flux vector $\mathbf{F}^*$. In this paper, a two-step method is used to approximate it.

In the first step, we consider a fluid particle occupies cell i bounded by two faces $i + 1/2$ and $i - 1/2$. The conservative quantities of this particle satisfies

$$\tilde{\Omega}_i \tilde{\mathbf{U}}_i = \Omega_i \mathbf{U}_i^n - \Delta t(\mathbf{P}^*_{i+1/2} - \mathbf{P}^*_{i-1/2}), \tag{4}$$

where the tilded variables represent the updated states of the particle, or the solutions in the Lagrangian frame. Notice that the flux vector is $\mathbf{P}$ instead of $\mathbf{F}$. Because of the motion of the particle, it has been advected away from the original Eulerian cell shown in Fig.1. In the second step, the conservative quantities in the Eulerian cell i is found by remapping the Lagrangian solution. Suppose the solution is piecewise constant, conservative quantities, $\Omega_i \mathbf{U}_i^{n+1}$, is the sum of two portions, AB and BC, as shown in Fig.1,

$$\Omega_i \mathbf{U}_i^{n+1} = (u^*_{i-1/2}\Delta t)\tilde{\mathbf{U}}_{i-1} + (\tilde{\Omega}_i - u^*_{i+1/2}\Delta t)\tilde{\mathbf{U}}_i. \quad (5)$$

Substituting (4) into (5), we get

$$\Omega_i \mathbf{U}_i^{n+1} = \Omega_i \mathbf{U}_i^n - \Delta t(\tilde{\mathbf{U}}_i u^*_{i+1/2} + \mathbf{P}^*_{i+1/2} - \tilde{\mathbf{U}}_{i-1} u^*_{i-1/2} - \mathbf{P}^*_{i-1/2}). \quad (6)$$

Now, it is clear that the two-step procedure leads to flux vector,

$$\mathbf{F}^* = \mathbf{U}^*_{i+1/2} u^*_{i+1/2} + \mathbf{P}^*_{i+1/2}, \quad (7)$$

where

$$\mathbf{U}^*_{i+1/2} = \begin{cases} \tilde{\mathbf{U}}_i & \text{for} \quad u^*_{i+1/2} \geq 0, \\ \tilde{\mathbf{U}}_{i+1} & \text{for} \quad u^*_{i+1/2} < 0. \end{cases} \quad (8)$$

The flux vector relies solely on the estimate of velocity and pressure at interfaces, which is much simpler than most Riemann solver that can resolve a stationary contact. We adopt the linear acoustic solution for the velocity and pressure. If the piecewise linear distribution is assumed instead, similarly to the MUSCL method, the two-step method can be readily extended to second-order accuracy. Fig. 2 gives the first-order and the second-order results of the Sod's shock tube problem. More details will be reported in full paper.

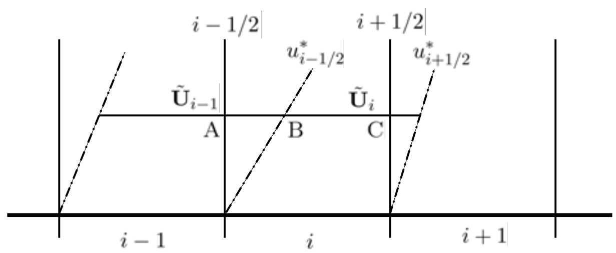

Figure 1: Construction of the two-step Riemann solver

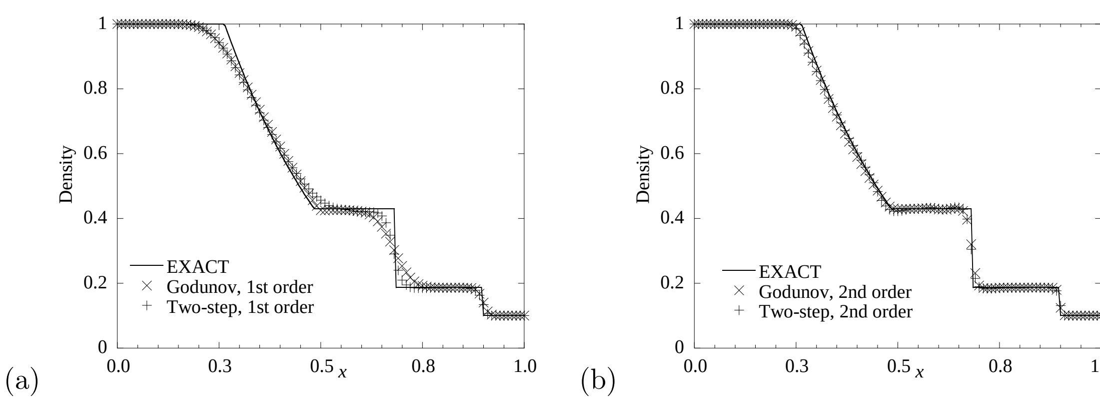

Figure 2: Shock tube problem: (a) first order scheme, (b) second-order scheme. The two-step Riemann solver is compared with the exact Godunov solver.

References

[1] Després B, Lagoutière F, Contact discontinuity captuing schemes for linear advection and compressible gas dynamics, J of Scientific Computing, **16** (2001) 479-524