Efficient Implementation of the CPR Formulation for the Navier-Stokes Equations on GPUs

Malte Hoffmann*, Claus-Dieter Munz* and Z. J. Wang**
Corresponding author: Malte.Hoffmann@gmx.de

* Institute für Aerodynamik und Gasdynamik, Universität Stuttgart
Pfaffenwaldring 21, 70569, Germany
** Department of Aerospace Engineering and CFD Center, Iowa State University
50011 Ames, USA

Abstract: The correction procedure via reconstruction (CPR) formulation for the Euler and Navier-Stokes equations is implemented on a graphics processing unit (GPU) with both explicit and implicit time-stepping schemes. For the implicit time integration, a modified non-linear lower-upper symmetric Gauss-Seidel (LU-SGS) approach is implemented and for the explicit time-stepping a 3-stage Runge-Kutta scheme is used. A speed-up factor of up to two orders of magnitude has been demonstrated.

Keywords: GPU Computing, Euler and Navier-Stokes equations, 2D unstructured triangular grids, Implicit time integration, Double precision, Correction procedure via reconstruction, High-Order.

1 Introduction

Over the past few years GPU computing has become widespread in the CFD community. Adaptive high-order methods are scalable and local, and are ideal suited for the GPU architecture. Most of the previous studies used explicit time integration and single precision floating point operations. The present study introduces an implicit LU-SGS scheme for steady state problems on GPUs and calculations with double precision floating point operations. The CPR method [1] is used to solve the Euler and Navier-Stokes equations on 2D unstructured triangular grids.

2 Implicit Time Stepping on a GPU

The equation for the implicit LU-SGS scheme [2] for steady state problems can be written as

$$\left(\frac{I}{\Delta t} - \frac{\partial \mathrm{Res}_i}{\partial Q_i}\right)\left(Q_i^{(w+1)} - Q_i^{(w)}\right) = \mathrm{Res}_i(Q^*), \tag{1}$$

where I is the identity matrix, Δt the current time step, Res_i the residual vector for cell i, Q_i the solution vector for cell i, w a iteration index and superscript $*$ stands for the most recent updated solution. The cell matrix $D_i = \partial \mathrm{Res}_i / \partial Q_i$ can be computed numerical as showed in Ref. [2]. For a CPU version, Eq. (1) can be solved with a symmetric forward and backward sweep over all cells. For the GPU version however the equation is modified in the following way

$$Q_i^{(w+1)} = D_i^{-1}\mathrm{Res}_i(Q_i, Q_{nb}^*) + Q_i^{(w)}, \tag{2}$$

where subscript nb stands for neighboring cells of cell i. Cell coloring is employed to parallelize the LU-SGS algorithm on GPUs. With the *Four Color Theorem*[3] an unstructured mesh (which can be transformed into a planar graph or map) can be colored with at most four colors in a way that no neighboring cells sharing the same color. By doing this, cells with the same color can be updated in parallel on a GPU, independently from other colors.

Introducing four colors *red,yellow,blue* and *green*, the symmetric forward and backward sweep can be written together as

$$Red \rightarrow Yellow \rightarrow Blue \rightarrow Green \rightarrow Green \rightarrow Blue \rightarrow Yellow \rightarrow Red.$$

A four colored mesh can be seen in Fig. 1.

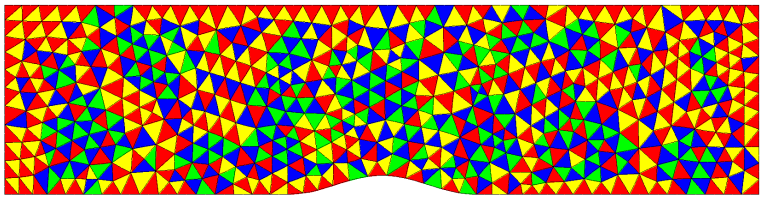

Figure 1: 2D triangular mesh colored with four colors (994 cells: 313 Red, 257 Yellow, 252 Blue and 172 Green)

3 Results

For polynomial degrees $k = 1 - 5$, the maximum speed-ups for the implementations are shown in Fig. 2. The calculations on the GPU are carried out with double precision floating point operation on one NVIDIA® Tesla™ C2050 and are compared to a similar code on one core of a Intel® Xeon® CPU (X5650@2.67GHz) running the same simulations. For the explicit scheme over two orders of magnitude speed-up has been achieved, while for the implicit scheme a speed-up factor up to 89 has been obtained.

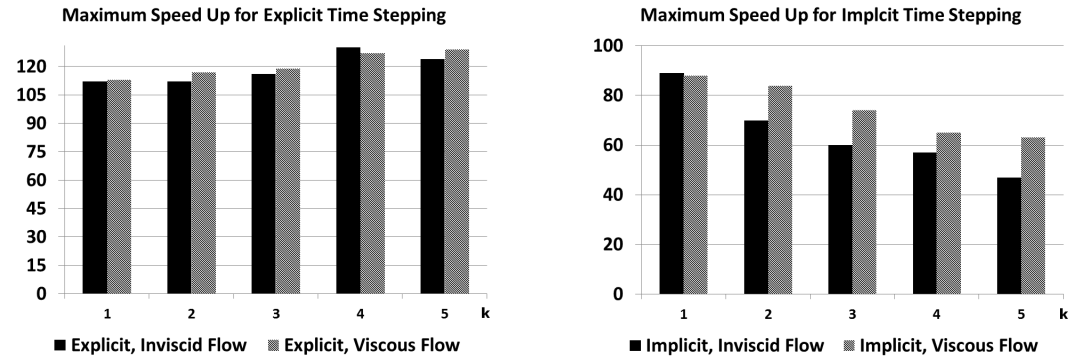

Figure 2: Maximum Speed Up for Explicit and Implicit Time Stepping Schemes

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