Numerical Simulation of Incompressible Flow using a Velocity-Pressure-Vorticity Formulation

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Abstract: The two-dimensional incompressible flow equations are solved by iterating on the kinematics, pressure and momentum equations producing a solution that satisfies mass and vorticity definitions up to machine accuracy. Given a vorticity and mass fields, a functional is constructed to measure the kinematic imbalance of a velocity field. The velocity equations are produced by minimizing a discrete functional, subject to constrains imposed by boundary conditions. A suitable preconditioning and interpolation technique are used to balance precision and computation speed. The Poisson equation for pressure is solved similarly by minimizing a suitable functional. The momentum equations are then solved using a finite volume approach. A controlled amount of artificial viscosity is added according to mesh size and Reynolds number, resulting in a stable calculation.

Keywords: Numerical Algorithms, Computational Fluid Dynamics, Vorticity, Discrete Functional, Minimization.

1. Introduction

Classically, formulations of the incompressible Navier-Stokes equations using a scalar stream function and vorticity are computationally attractive and conserve mass automatically but generalization to three dimensional flows are nontrivial [1]. Other techniques like velocity-vorticity formulations aim to simplify the gap between two and three dimensions, but they impose mass conservation explicitly as an extra equation. New challenges arise with these techniques, such as chessboard decoupling of the velocity equations and complications in multiply-connected domains [2], [3], [4]. Velocity-pressure methods have proven practical as in [5], but they rely on a pressure and velocity correction. Moreover, most incompressible flow solvers do not guarantee that mass and vorticity definitions are preserved in the discrete sense. Consequently, their solutions are corrupted by small amounts of mass and vorticity generated in the flow field.

The present work develops a new approach using a velocity-pressure-vorticity formulation. A Poisson equation for pressure is solved, so no pressure correction or artificial compressibility is necessary. Further techniques deal with velocity decoupling and to accelerate convergence, leading to a one-parameter family of schemes that balance decoupling precision and convergence speed depending on the geometry. Similarly, the vorticity is obtained from the momentum equation. Variable up-winding can be tuned to balance stability and true viscosity. The technique is presented for steady two dimensional flows, but it can be generalized to unsteady three-dimensional and compressible flows.

2. Formulation

Separate formulations are constructed for the kinematics, pressure and momentum equations. After discretization, a solution method is implemented alternating velocity, pressure and vorticity iterations in a segregated manner. On every iteration, each separate solution field is solved independently using the PARDISO direct solver. The three equations to be solved are described below,

In the differential level, given a fixed mass s and vorticity $\boldsymbol{\omega}$ fields on a volume Ω, the kinematic velocity field $\boldsymbol{V}$ can be obtained through calculus of variations by minimizing the functional;

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$$E = \iint_{\Omega} (\nabla \cdot \boldsymbol{V} - s)^2 + \|\nabla \times \boldsymbol{V} - \boldsymbol{\omega}\|^2,$$

subject to suitable boundary conditions, such as no-penetration and some far-field information. A discrete version of the functional E is constructed. Further analysis shows that the discrete terms must be preconditioned by the inverse of a discrete cell area in order for it to yield a correct Laplace formula. Moreover, chessboard decoupling is avoided by introducing a one-parameter family of discrete radial interpolation schemes that balances decoupling precision and convergence speed. The kinematic equations are obtained through a Lagrange multipliers technique minimizing the discrete functional with respect to the velocity values, constrained by the boundary conditions.

Given a velocity field $\boldsymbol{V}$, a quadratic functional is built to measure the pressure imbalance;

$$F = \iint_{\Omega} \|\nabla \mathrm{P} - \boldsymbol{f}\|^2,$$

where the vector term $\boldsymbol{f}$ is computed from the momentum equation (below) and simplified using the mass conservation equation. Preconditioning and radial interpolation schemes are again implemented to improve performance. Algebraic minimization again yields the discrete Poisson pressure equation.

Given an advection velocity field $\boldsymbol{V}$ and a pressure field P, the vorticity equation is typically obtained by taking the curl of the vector form of the momentum equation;

$$\nabla \cdot \left(\boldsymbol{V} \otimes \overline{\boldsymbol{V}}\right) = -\nabla P + \frac{1}{\mathrm{Re}} \nabla^2 \overline{\boldsymbol{V}}.$$

However, such techniques may compromise the robustness of the solution, especially in terms of its dependence from the boundary geometry. For that reason, the full momentum equation is instead discretized using a direct finite volume technique and solved for the velocity $\overline{\boldsymbol{V}}$. Furthermore, variable up-winding is applied to balance stability and true viscosity. The vorticity is computed a posteriori.

3. Conclusion and Future Work

The formulation was applied to several flow problems and proved to be a promising technique for numerical simulations of incompressible flow (see Figure 1). In most cases convergence to machine accuracy was achieved in a few dozen iterations and requiring usually 10 to 30 seconds of compute time in a standard PC architecture. It was verified that the mass and vorticity definitions discrete balance was satisfied to round-off error and that the accuracy of the method depends only on the grid. Boundary singularities that lead to numerical chessboard pattern solutions are remedied by use of the radial interpolation scheme. More results including flows over airfoils will be presented in the paper.

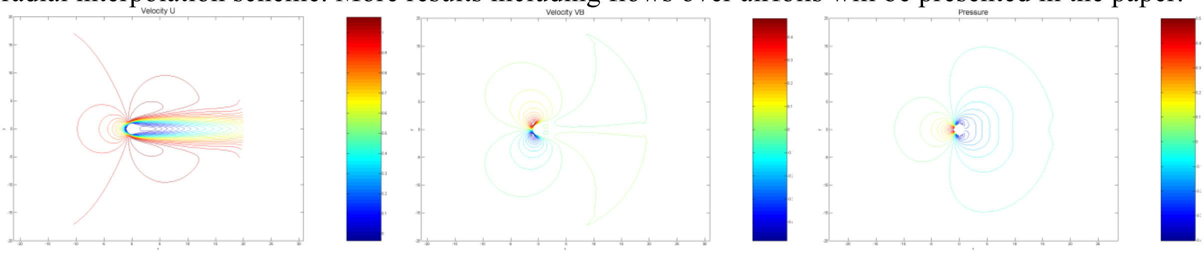

Figure 1. Velocities and pressure around a cylinder at Re=40.

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