New Explicit Runge-Kutta Methods for the Incompressible Navier-Stokes Equations

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Abstract: New explicit Runge-Kutta methods are presented for time integration of the incompressible Navier-Stokes equations. These methods are high-order accurate for both velocity and pressure, even when the boundary conditions or the mesh are time-dependent. Computations for an actuator disk in a time-dependent inflow support the correctness of the analytically derived methods.

Keywords: Incompressible Navier-Stokes, Time integration, Temporal accuracy, Runge-Kutta.

1 Introduction

Time integration of the incompressible Navier-Stokes equations with Runge-Kutta methods is not straightforward due to the differential-algebraic nature of the equations. In this work we investigate the temporal order of accuracy of velocity and pressure when explicit Runge-Kutta methods are applied to the incompressible Navier-Stokes equations.

It turns out that both velocity and pressure can be computed to the classical ('ODE') order of accuracy, except in two important cases [1]. First, if the boundary conditions for the normal velocity component depend on time, the order of accuracy of the pressure is affected. Second, if the mesh is time-dependent, the order of accuracy of both velocity and pressure is affected. We propose a number of new methods for these cases that are third- and fourth-order for the velocity and second order for the pressure. The second-order accuracy of the pressure is obtained with a new technique based on reconstruction of instantaneous pressure values from time-averaged values.

2 Computational results

To illustrate the effect of time-dependent boundary conditions, we compute the laminar flow through an actuator disk with unsteady inflow conditions. This is a simplified model of a wind turbine experiencing a time-varying wind field. The inflow conditions are given by $u_{\text{inflow}}(t) = \cos\alpha(t)$, $v_{\text{inflow}}(t) = \sin\alpha(t)$, where $\alpha(t) = \frac{\pi}{6}\sin(t/2)$, i.e., an inflow with constant magnitude but changing direction.

The flow field at $t = 4\pi$ is shown in figure 1. The temporal error in velocity and pressure is computed by subtracting the flow field obtained from a simulation with very small time step. The resulting convergence of the velocity and pressure errors is shown in figure 2, for methods with three (S3) and four (S4) stages. The velocity attains its classical third-order accuracy for the three-stage (S3) method, and fourth-order for the four-stage (S4) method. Without appropriate measures the pressure is only first-order accurate (S3p1). With our new method the pressure is second accurate (S3p2 and S4p2). Note that in principle the pressure can be computed to the same order as the velocity if an additional Poisson solve is performed and if the boundary conditions are differentiable in time. Our approach does not require any significant additional computational effort or differentiability of boundary conditions, is second-order accurate and starts with a small error already at the largest time step considered.

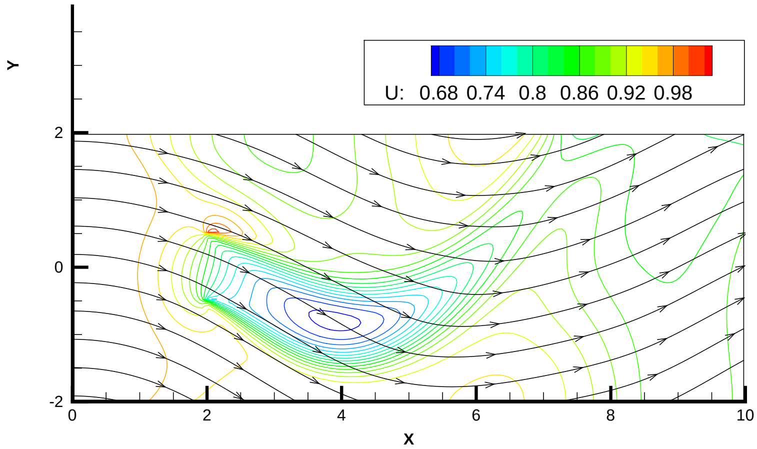

Figure 1: Flow through an actuator disk. Streamlines and u-contour lines at $t = 4\pi$.

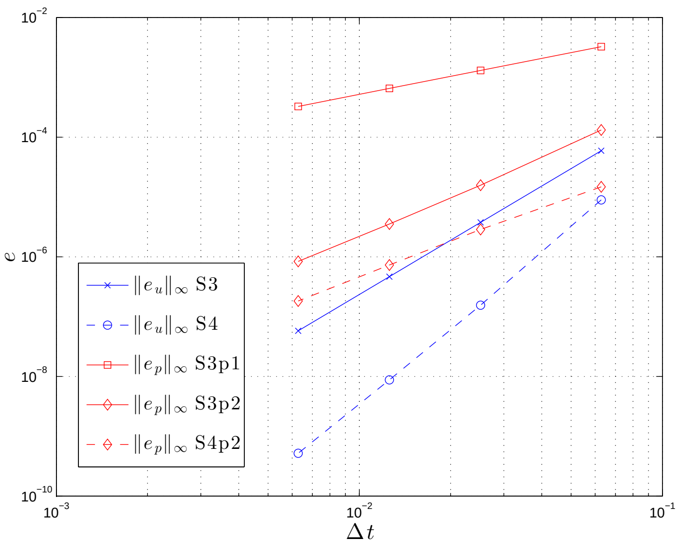

Figure 2: Velocity and pressure error at $t = 4\pi$ for a three- and four-stage Runge-Kutta method.

References

[1] B. Sanderse and B. Koren. Accuracy analysis of explicit Runge-Kutta methods applied to the incompressible Navier-Stokes equations. *J. Comput. Phys.*, 2011. doi: 10.1016/j.jcp.2011.11.028.