Oral presentation | Numerical methods
Numerical methods-IV
Wed. Jul 17, 2024 2:00 PM - 4:00 PM Room A

[8-A-04] A Third-order Weighted Compact Least-Squares Scheme for Hyperbolic Conservation Laws on Non-Uniform Grids

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Keywords: Weighted Compact Least-Squres, WENO, Non-uniform Grids

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A Third-Order Weighted Compact Least-squares Scheme for Hyperbolic Conservation Laws on Non-Uniform Grids

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BACKGROUND

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WENO Liu, Xu-Dong et al. (1994)

- based on point values
- based on polynomials Levy et al. CWENO (1999)

optimal spectral characteristic

optimal weights M.P. Martı´n et al. (2006) … Castro et al. (2011) …

Compact WENO Scheme

(based on point values)

- high-resolution in implicit scheme

Yu-Xin Ren et al. (2003) …

Weighted Compact Least-squares Scheme (WCLS) Wang, Qian et al. (2016)

- compact WENO scheme (based on polynomials)
- explicit polynomials ·
- High-Resolution :
- Robustness ∴

· compact least-squares reconstruction

: linear weights w_1, w_2

∴ non-linear weights

OUTLINE

- Weighted least-squares reconstruction
- Dispersion and dissipation analysis
- Optimal non-linear weights
- Extension to Euler equation
- Numerical tests
- Conclusions

RECONSTRUCTION

- Reconstruction polynomial

$$P_i(x) = \bar{u}_i + \sum_{j=1}^{k} a_{i,j}\varphi_i^j(x)$$

$k=2$

- 3-order polynomial

$$P_i(x) = \bar{u}_i + a_{i,1}\left(\frac{x-x_i}{h_i}\right) + a_{i,2}\left[\left(\frac{x-x_i}{h_i}\right)^2 - \frac{1}{12}\right]$$

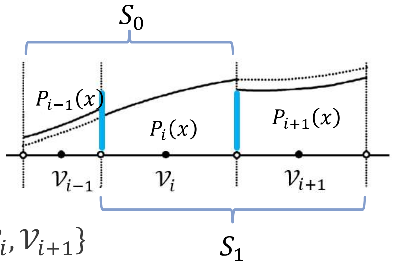
$\boxed{\text{unknowns } a_{i,1}, a_{i,2}}$

- Compact stencil

$S_i \coloneqq \{\mathcal{V}_{i-1}, \mathcal{V}_i, \mathcal{V}_{i+1}\}$

sub-stencil

$S_0 \coloneqq \{\mathcal{V}_{i-1}, \mathcal{V}_i\}, S_1 \coloneqq \{\mathcal{V}_i, \mathcal{V}_{i+1}\}$



- Determine unknowns

Minimize the point-value difference at the interface of neighboring cells (including derivatives):

$$P_i^{(o)}\left(x_{i\pm\frac{1}{2}}\right) = P_{i\pm1}^{(o)}\left(x_{i\pm\frac{1}{2}}\right), \quad o = 0,1 \dots n$$

$$n \le k = 2$$

$n = 2$

Each sub-stencil introduces 3 eqs.

The control volume $\mathcal{V}_i$ corresponding to 6 eqs.

RECONSTRUCTION

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Matrix form $\boldsymbol{E}_i\vec{a}_i + \boldsymbol{F}_i\vec{a}_{i-1} + \boldsymbol{G}_i\vec{a}_{i+1} = \vec{b}_i$

$$P_i(x_{i-\frac{1}{2}}) = P_{i-1}(x_{i-\frac{1}{2}})$$

S_0 → $$w_{i,1}\left[P_i^{(1)}(x_{i-\frac{1}{2}}) = P_{i-1}^{(1)}(x_{i-\frac{1}{2}})\right]$$

$$w_{i,2}\left[P_i^{(2)}(x_{i-\frac{1}{2}}) = P_{i-1}^{(2)}(x_{i-\frac{1}{2}})\right]$$

$\boldsymbol{w_{i,l} = w_l h^l}$

$$P_i(x_{i+\frac{1}{2}}) = P_{i+1}(x_{i+\frac{1}{2}})$$

S_1 → $$w_{i,1}\left[P_i^{(1)}(x_{i+\frac{1}{2}}) = P_{i+1}^{(1)}(x_{i+\frac{1}{2}})\right]$$

$$w_{i,2}\left[P_i^{(2)}(x_{i+\frac{1}{2}}) = P_{i+1}^{(2)}(x_{i+\frac{1}{2}})\right]$$

Least-squares $\boldsymbol{E}_i^T$

- block-tridiagonal system

$$\boldsymbol{D}_i\vec{a}_i + \boldsymbol{L}_i\vec{a}_{i-1} + \boldsymbol{U}_i\vec{a}_{i+1} = \vec{c}_i \quad i = 0,1\ldots N$$

- $\boldsymbol{M}\vec{A} = \vec{V}$

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{D}_1 & \boldsymbol{U}_1 & & & & \boldsymbol{L}_1 \\ \boldsymbol{L}_2 & \boldsymbol{D}_2 & \boldsymbol{U}_2 & & & \\ & \boldsymbol{L}_3 & \boldsymbol{D}_3 & \boldsymbol{U}_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ \boldsymbol{L}_N & & & & \boldsymbol{D}_N & \boldsymbol{U}_N \end{pmatrix}$$

$$\boxed{\vec{A} = (\vec{a}_1^T, \vec{a}_2^T \ldots \vec{a}_N^T)^T}$$

Explicit 3-order polynomial $P_i(x)$

OUTLINE

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SPECTRAL CHARACTERISTICS

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- Finite Fourier discrete
  - $u(x,t) = A_m(t)e^{ik_m x}$
  - $a_{i,1} = B_m(t)e^{ik_m x_i}$
    $a_{i,2} = C_m(t)e^{ik_m x_i}$
  - $\bar{u}(x,t) = \frac{A_m}{ik_m h}\left(e^{ik_m x_{i+0.5}} - e^{ik_m x_{i-0.5}}\right)$
- Modified wavenumber $\kappa' = \kappa \cdot \frac{\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}}{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}$

$$\frac{\hat{f}_i(x)}{a} = \bar{u}_i + a_{i,1}\left(\frac{x-x_i}{h_i}\right) + a_{i,2}\left[\left(\frac{x-x_i}{h_i}\right)^2 - \frac{1}{12}\right]$$

dispersion
- $Re(\kappa') = \frac{-6\sin\kappa\{n_1 + n_2\cos\kappa + n_3\cos 2\kappa\}}{d_1 + d_2\cos\kappa + d_3\cos 2\kappa}$

dissipation
- $Im(\kappa') = \frac{576 w_1^2 w_2^2\left[\sin\left(\frac{\kappa}{2}\right)\right]^6}{d_1 + d_2\cos\kappa + d_3\cos 2\kappa}$

$$n_1 = 2(w_1^2 + 3w_2^2 + 9w_1^2 w_2^2)$$
$$n_2 = w_1^2 - 6(1 + 4w_1^2)w_2^2$$
$$n_3 = 6w_1^2 w_2^2$$
$$d_1 = -9(w_1^2 + w_2^2 + 12w_1^2 w_2^2)$$
$$d_2 = 8w_1^2(-1 + 18w_2^2)$$
$$d_3 = -[w_1^2 + 9(-1 + 4w_1^2)w_2^2]$$

w_1

w_2

SPECTRAL CHARACTERISTICS

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- High-resolution

$$I = \int_0^\pi e^{\nu(\pi-\kappa)}\left(\sigma[Re(\kappa') - \kappa]^2 + (1-\sigma)\left[Im(\kappa') - \gamma\left(\sin\frac{\kappa}{2}\right)^\mu\right]^2\right)d\kappa.$$

$$\nu = 7, \mu = 16 \qquad \sigma = 0.5$$
$$W_1 = 0.0111760 \qquad W_2 = 0.00091551$$
$$\nu = 7, \mu = 8 \qquad \sigma = 0.1$$
$$W_1 = 0.0746201 \qquad W_2 = 0.00863398$$
$$\nu = 8, \mu = 8 \qquad \sigma = 0.1$$
$$W_1 = 0.0787969 \qquad W_2 = 0.00802832$$

M.P. Martı´n et al.
Journal of Computational Physics (2006)

SPECTRAL CHARACTERISTICS

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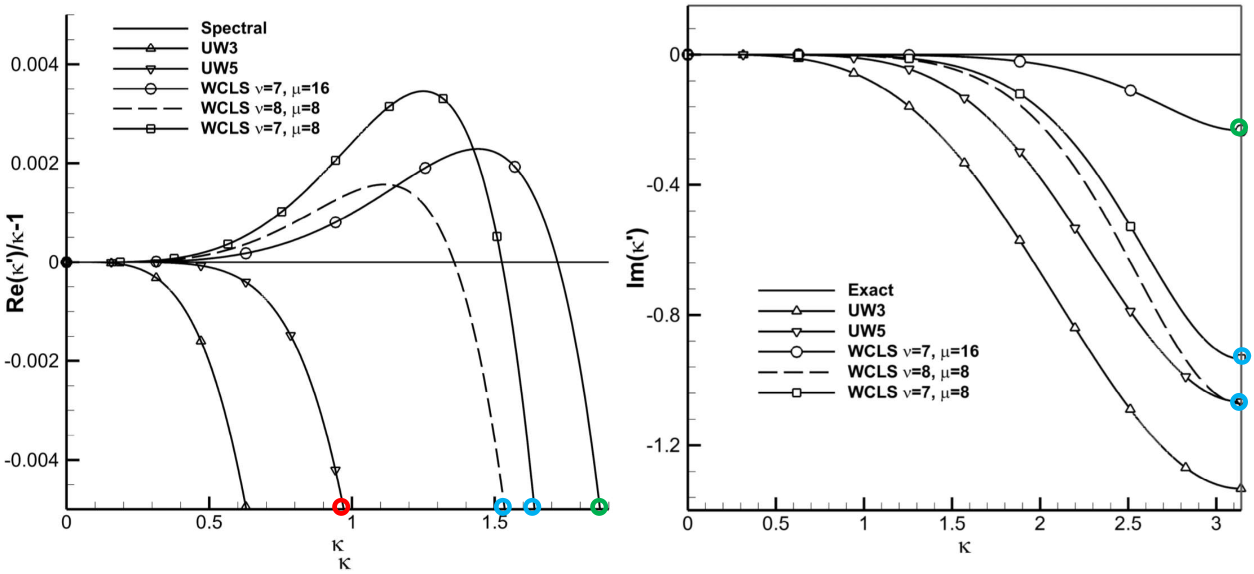
linear scheme



$|Re(\kappa')/\kappa - 1| \leq 0.5\%$

Wang, Qian et al. Journal of Computational Physics (2016)

OUTLINE

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NON-LINEAR WEIGHTS

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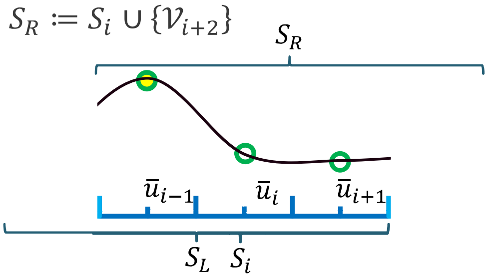
- Critical points
  - Local extreme points
  - discontinuities
  - …… On the accuracy loss of 3-order WENO schemes
- Over dissipation
- Extended stencils

$S_L := S_i \cup \{\mathcal{V}_{i-2}\}$

$S_R := S_i \cup \{\mathcal{V}_{i+2}\}$



- Smooth indicator

$$w_L \cdot \left\{ \begin{aligned} & P_i(x_{i-\frac{1}{2}}) = P_{i-1}(x_{i-\frac{1}{2}}) \\ & w_{i,1}\left[P_i^{(1)}(x_{i-\frac{1}{2}}) = P_{i-1}^{(1)}(x_{i-\frac{1}{2}})\right] \\ & w_{i,2}\left[P_i^{(2)}(x_{i-\frac{1}{2}}) = P_{i-1}^{(2)}(x_{i-\frac{1}{2}})\right] \end{aligned} \right\} \quad w_R \cdot \left\{ \begin{aligned} & P_i(x_{i+\frac{1}{2}}) = P_{i+1}(x_{i+\frac{1}{2}}) \\ & w_{i,1}\left[P_i^{(1)}(x_{i+\frac{1}{2}}) = P_{i+1}^{(1)}(x_{i+\frac{1}{2}})\right] \\ & w_{i,2}\left[P_i^{(2)}(x_{i+\frac{1}{2}}) = P_{i+1}^{(2)}(x_{i+\frac{1}{2}})\right] \end{aligned} \right\}$$

Least-squares $\boldsymbol{D}_i\vec{a}_i + \boldsymbol{L}_i\vec{a}_{i-1} + \boldsymbol{U}_i\vec{a}_{i+1} = \vec{c}_i$

optimal weight

- Non-linear weights w_L, w_R
  - If S_i is smooth, both of w_L, w_R approach to 1.
  - If a discontinuity crosses S_i, w_L or w_R approaches to 0.

Baeza et al. (2020)
SIAM Journal on Scientific Computing

NON-LINEAR WEIGHTS

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- OWENO3

Baeza et al. SIAM Journal on Scientific Computing (2020)

optimal weight

- Non-linear weight $w_0 := \beta * c_0 + (1-\beta)\widetilde{w}_0$

non-linear part $\dfrac{I_1+\varepsilon}{I_0+I_1+\varepsilon}$

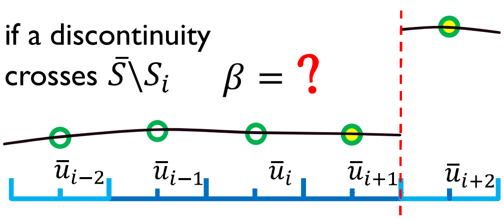
- Thresholds $\beta = \dfrac{J}{J+\tau+\varepsilon}$

Case1

$$\beta = \begin{cases} 1 + \mathcal{O}(h^{4-2k}) + \mathcal{O}(\varepsilon) & \text{if } \bar{S} \text{ is smooth, } f \in \mathcal{C}^3, \\ \mathcal{O}(h^2) + \mathcal{O}(\varepsilon) & \text{if a discontinuity crosses } S.\ S_i. \end{cases}$$

Case2

if a discontinuity crosses $\bar{S}\backslash S_i$ $\quad \beta = ?$



- Case3: $I_2 = \mathrm{O}(1) = (\bar{u}_{i+2} - \bar{u}_{i+1})^2$

$$\beta = \frac{J}{J+\tau+\varepsilon} = \frac{1}{1+\frac{\tau}{J}} - \mathrm{O}(\varepsilon)$$

$$= \frac{1}{1+\frac{\bar{\mathrm{O}}(1)}{\bar{\mathrm{O}}(h^{2m})}} - \mathrm{O}(\varepsilon)$$

$$= \frac{1}{1+\bar{\mathrm{O}}(h^{-2m})} - \mathrm{O}(\varepsilon)$$

$$= \bar{\mathrm{O}}(h^{2m}) + \mathrm{O}(\varepsilon)$$

- Though S_i is smooth, the optimal weight isn't dominate.

Whether the non-linear weight w_0 can still approach to optimal weight?

NON-LINEAR WEIGHTS

- Non-linear weights (WCLS)

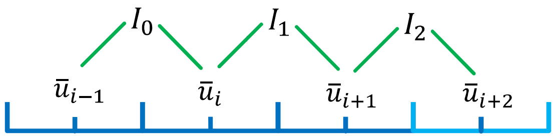
  $w_R = \beta * 1 + (1-\beta)\widetilde{w}_R$

- Optimal
  - $\widetilde{w}_R = \frac{2[I_0+\varepsilon(h)]+I_2}{I_0+I_1+2\varepsilon(h)+I_2} \in (0,2]$

  If $I_2 = \overline{O}(1)$ → $\widetilde{w}_R = 1 + O(h^2)$, $\beta = \overline{O}(h^2) + O(\varepsilon)$

  → $w_R = \boxed{1} + O(h^2)$

  - $J_R = I_0 I_1 + I_0 I_2 + I_1 I_2$



- Characteristics

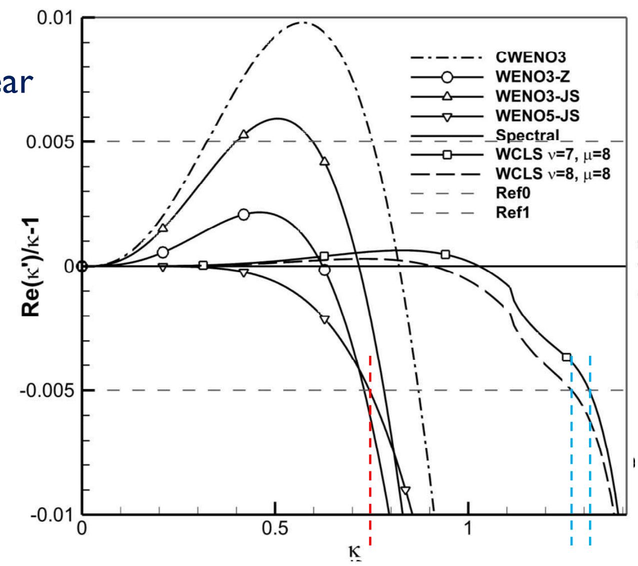
  - $$\beta = \begin{cases} 1 - O(h^{4-2k}) - O(\varepsilon) & S_R\text{光滑}, \\ \overline{O}(h^{2m}) + O(\varepsilon) & S_i\text{有间断}, \\ \overline{O}(h^2) + O(\varepsilon) & I_2 = \overline{O}(1). \end{cases}$$

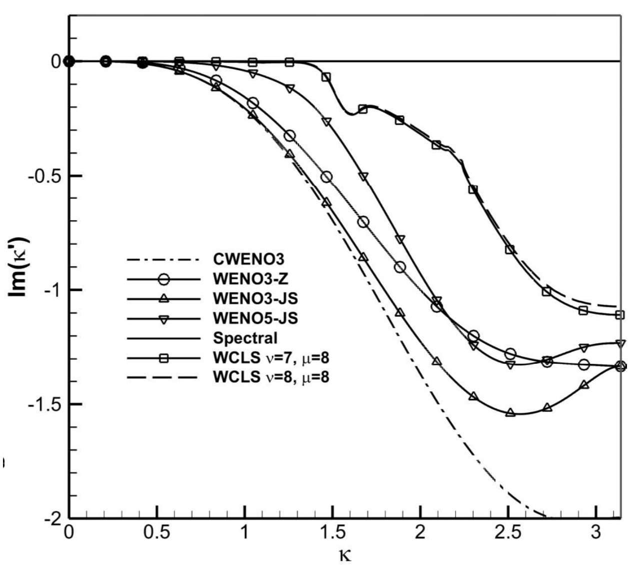
  - $$w_R = \begin{cases} 1 + O\left(h^{4-2k}\right) + O(\varepsilon) & S_R\text{光滑}, \\ \widetilde{w}_R + O(h^2) + O(\varepsilon) & S_i\text{有间断}, \\ 1 + O(h^2) + O(\varepsilon) & I_2 = \overline{O}(1). \end{cases}$$

  - $w_R + w_L = 2 + O(h^2) + O(\varepsilon)$

NON-LINEAR WEIGHTS

Non-linear scheme





ADR

Sergio Pirozzoli Journal of Computational Physics (2006)

OUTLINE

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EULER EQUATION

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- **1D**-Euler equation

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}=0 \longrightarrow \frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{J}\frac{\partial \boldsymbol{U}}{\partial x}=0 \longrightarrow \boldsymbol{L}\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{\Lambda L}\frac{\partial \boldsymbol{U}}{\partial x}=0$$

conservative variables $\boldsymbol{U}=\begin{pmatrix}\rho\\ \rho u\\ \rho E\end{pmatrix}$ $\boldsymbol{F}=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho u H\end{pmatrix}$

$$\boldsymbol{\Omega}=\boldsymbol{L}\cdot\boldsymbol{U}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t}+\boldsymbol{\Lambda}\frac{\partial \boldsymbol{\Omega}}{\partial x}=0$$

- conservative variables $\boldsymbol{\Omega}$

$$\frac{\partial \omega^{(m)}}{\partial t}+\lambda_m(\overline{U})\frac{\partial \omega^{(m)}}{\partial x}=0$$

$$\omega_i^{(m)}=\overline{\omega}_i^{(m)}+\alpha_i^{(1,m)}\delta x_i+\alpha_i^{(2,m)}[(\delta x_i)^2-I_2]$$

$$m=1,2,3$$

- coefficients to be determined

$$\begin{bmatrix}\alpha_{i,l}^{(1)}\\ \alpha_{i,l}^{(2)}\\ \alpha_{i,l}^{(3)}\end{bmatrix}=\boldsymbol{L}\cdot\begin{bmatrix}a_{i,l}^{(1)}\\ a_{i,l}^{(2)}\\ a_{i,l}^{(3)}\end{bmatrix}$$

$$l=1,2$$

$$\begin{aligned}\vec{\omega}_i &= \vec{\overline{\omega}}_i+\boxed{\vec{\alpha}_{i,1}}\delta x_i+\boxed{\vec{\alpha}_{i,2}}[(\delta x_i)^2-I_2]\\ &=\boxed{\boldsymbol{L}}\cdot\underbrace{\{\vec{\overline{u}}_i+\boxed{\vec{a}_{i,1}}\delta x_i+\boxed{\vec{a}_{i,2}}[(\delta x_i)^2-I_2]\}}_{\boldsymbol{U}}\end{aligned}$$

EULER EQUATION

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$$\boldsymbol{\Omega}_i^{(o)}|_{x_{i\pm\frac{1}{2}}} = \boldsymbol{\Omega}_{i\pm1}^{(o)}|_{x_{i\pm\frac{1}{2}}}, \qquad o = 0,1,2$$

- block-tridiagonal system

$$\boldsymbol{L}_i\vec{\boldsymbol{A}}_{i-1} + \boldsymbol{D}_i\vec{\boldsymbol{A}}_i + \boldsymbol{U}_i\vec{\boldsymbol{A}}_{i+1} = \vec{\boldsymbol{B}}_i$$

6

$$i = 0,1 \ldots N$$

unknowns $\vec{\boldsymbol{A}}_i = \left(\vec{a}_{i,1}^T, \vec{a}_{i,2}^T\right)^T$

$$\vec{a}_{i,l}^T = \left(\alpha_{i,l}^{(1)}, \alpha_{i,l}^{(2)}, \alpha_{i,l}^{(3)}\right)^T$$

- $\boldsymbol{M}\vec{A} = \vec{V}$

$$\longrightarrow \boldsymbol{M} = \begin{pmatrix} \boldsymbol{D}_1 & \boldsymbol{U}_1 & & & & \\ \boldsymbol{L}_2 & \boldsymbol{D}_2 & \boldsymbol{U}_2 & & & \\ & \boldsymbol{L}_3 & \boldsymbol{D}_3 & \boldsymbol{U}_3 \cdot \boxed{\boldsymbol{L}} & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \boldsymbol{D}_N & \boldsymbol{U}_N \end{pmatrix}$$

$$\vec{\alpha}_{i,1} = \boxed{\boldsymbol{L}} \cdot \vec{a}_{i,1}$$
$$\vec{\alpha}_{i,2} = \boldsymbol{L} \cdot \vec{a}_{i,2}$$

explicit 3-order polynomial $U_i(x)$

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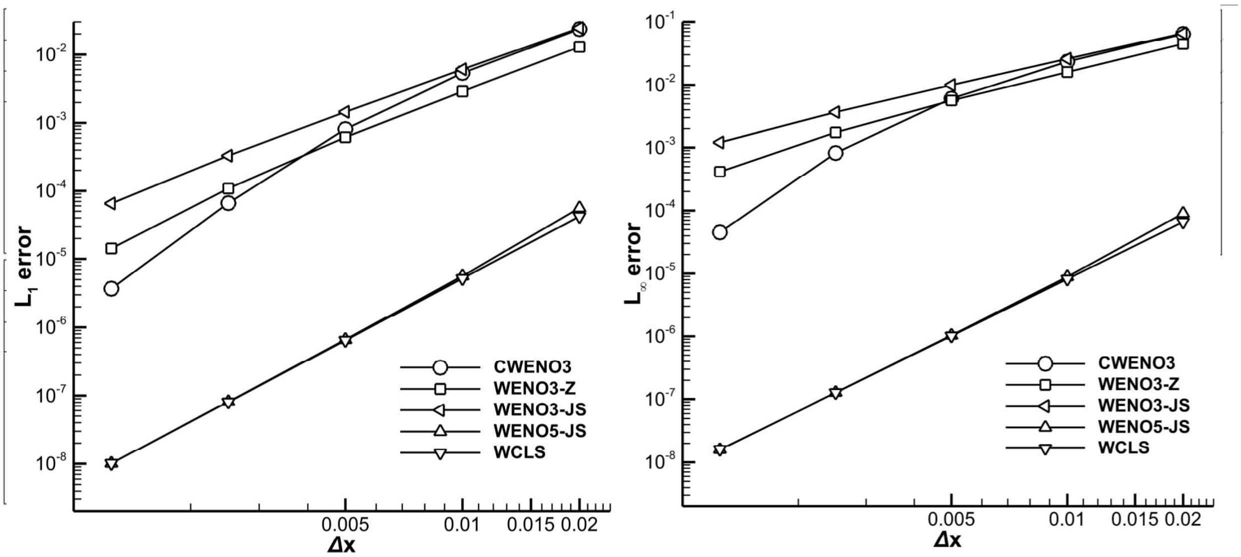
1 Order of the scheme



NUMERICAL RESULTS

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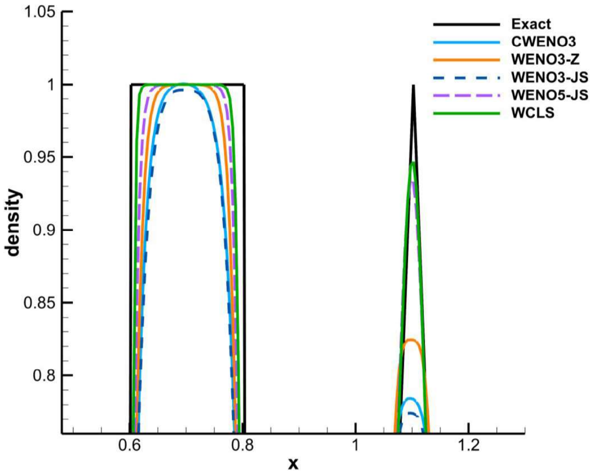
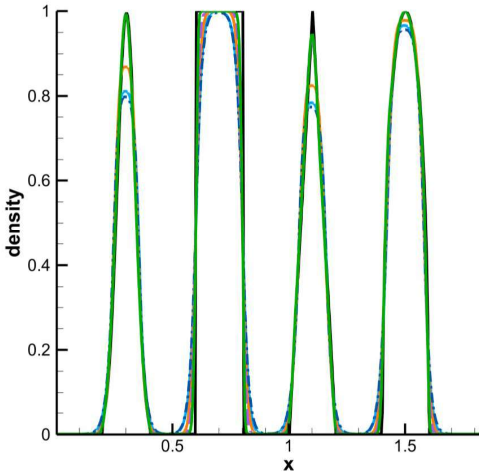
2

GSTE

N=400, , L=2
CFL=0.5, t=2





I.C.

$$u_0(x) = \begin{cases} \frac{1}{6}\left(G(x,\beta,z-\delta) + G(x,\beta,z+\delta) + 4G(x,\beta,z)\right) & \text{if } -0.8 < x < -0.6, \\ 1 & \text{if } -0.4 < x < -0.2, \\ 1 - |10(x-0.1)| & \text{if } 0.0 < x < 0.2, \\ \frac{1}{6}\left(F(x,\beta,a-\delta) + F(x,\beta,a+\delta) + 4F(x,\beta,a)\right) & \text{if } 0.4 < x < 0.6, \\ 0 & \text{otherwise,} \end{cases}$$

NUMERICAL RESULTS

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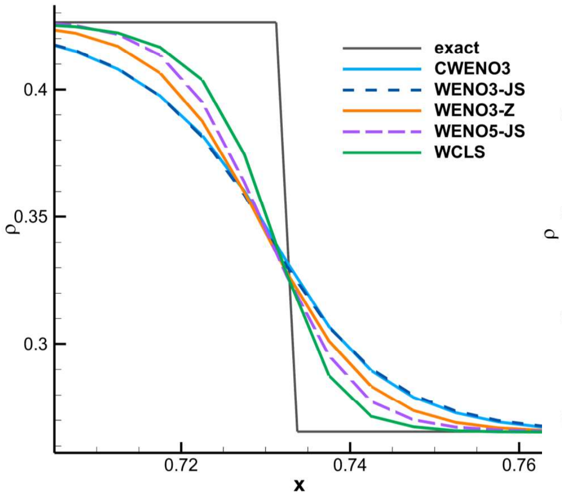
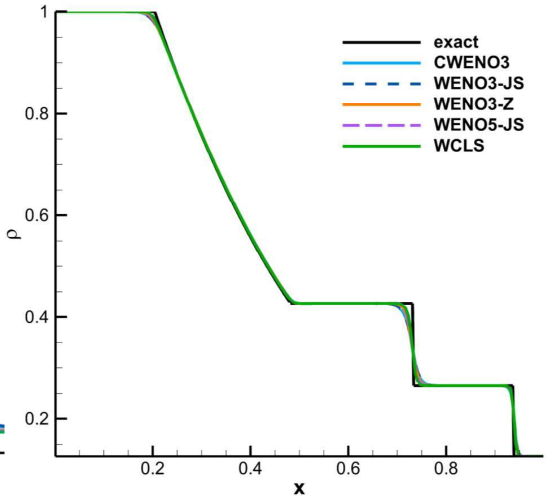
- Sod





N=200, L=1
CFL=0.5, t=0.25

I.C.

$$(\rho_0, u_0, p_0) = \begin{cases} (1, 0, 1) & \text{for } 0 \le x \le 0.5 \\ (0.125, 0, 0.1) & \text{otherwise} \end{cases} \quad (0 \le x \le 1).$$

NUMERICAL RESULTS

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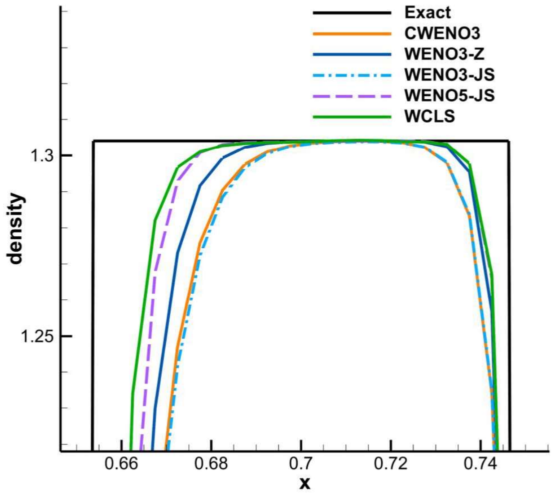
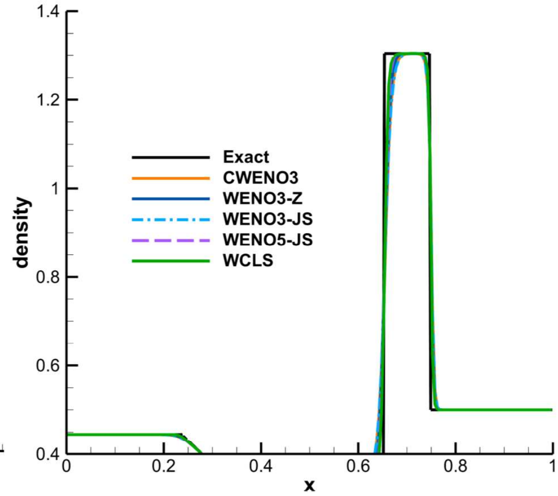
- Lax





N=200, L=1
CFL=0.5, t=0.1

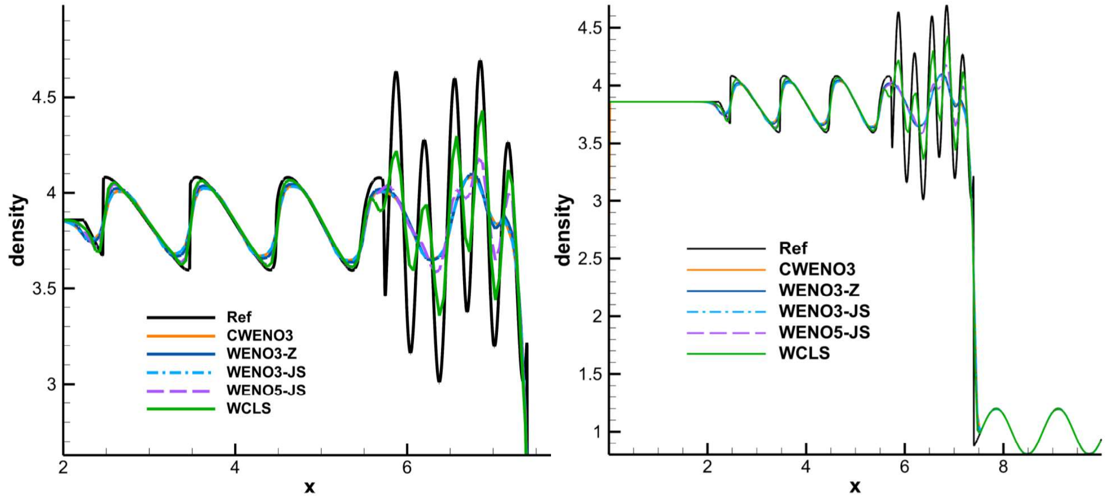
I.C.

$$(\rho_0, u_0, p_0) = \begin{cases} (0.445, 0.698, 3.528) & \text{for } 0 \le x \le 0.5 \\ (0.5, 0, 0.571) & \text{otherwise} \end{cases} \quad (0 \le x \le 1).$$

NUMERICAL RESULTS

Shu-Osher

N=200, L=10
CFL=0.5, t=1.8



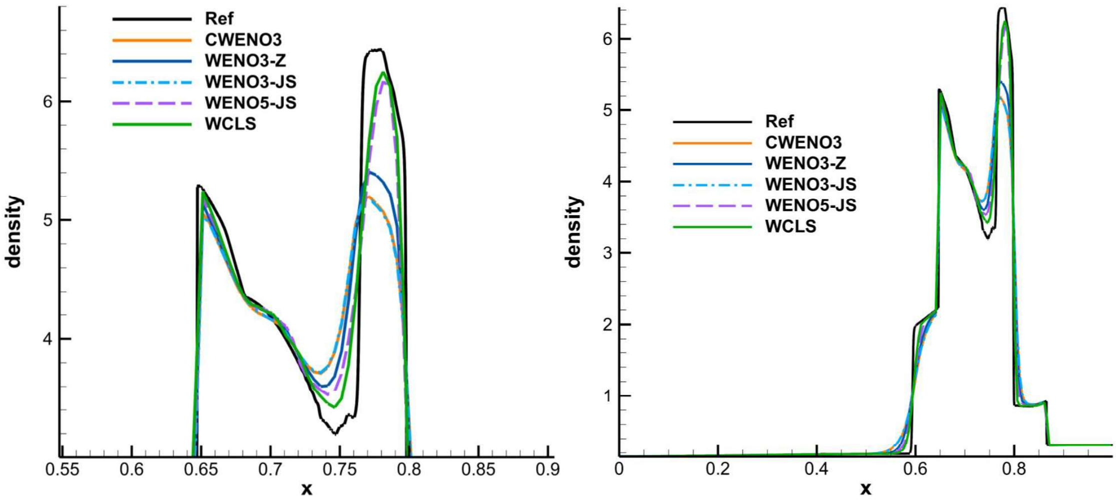
I.C.

$$(\rho_0, u_0, p_0) = \begin{cases} (3.857143, 2.629369, 10.33333) & \text{for } 0 \le x < 1 \\ (1 + 0.2\sin(5x), 0, 1) & \text{for } x \ge 1 \end{cases}$$

NUMERICAL RESULTS

Two interacting blast waves

N=400, L=1
CFL=1, t=0.038



I.C.

$$(\rho, u, p) = \begin{cases} (1, 0, 1000) & \text{for } 0 \le x \le 0.1 \\ (1, 0, 0.01) & \text{for } 0.1 < x \le 0.9 \\ (1, 0, 100) & \text{otherwise} \end{cases}$$

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Propagation of Broadband sound waves

N=128, L=1, k=12

CFL=0.2, t=$\frac{1}{1+\sqrt{1.4}}$

$$p(x,0) = p_0\left(1 + \varepsilon \sum_{k=1}^{N/2} (E_p(k))^{0.5} \sin(2\pi k(x+\psi_k))\right),$$

$$\rho(x,0) = \rho_0 (p(x,0)/p_0)^{1/\gamma},$$

$$u(x,0) = u_0 + \tfrac{2}{\gamma-1}(c(x,0)/c_0),$$

$$E_p(k) = \left(\frac{k}{k_0}\right)^4 e^{-2(k/k_0)^2}$$

I.C.