Oral presentation | Turbulent flow

Turbulent flow-II

Wed. Jul 17, 2024 10:45 AM - 12:45 PM Room B

[7-B-01] Ceps-Relambda scaling of inhomogeneous and unsteady turbulence

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Keywords: Taylor's dissipation law, Perturbation analysis, Kolmogorov flow

C_ϵ–Re_λ Scaling of Inhomogeneous and Unsteady Turbulence

12th International Conference on Computational Fluid Dynamics

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“Zeroth law” of turbulence Frisch (1995)

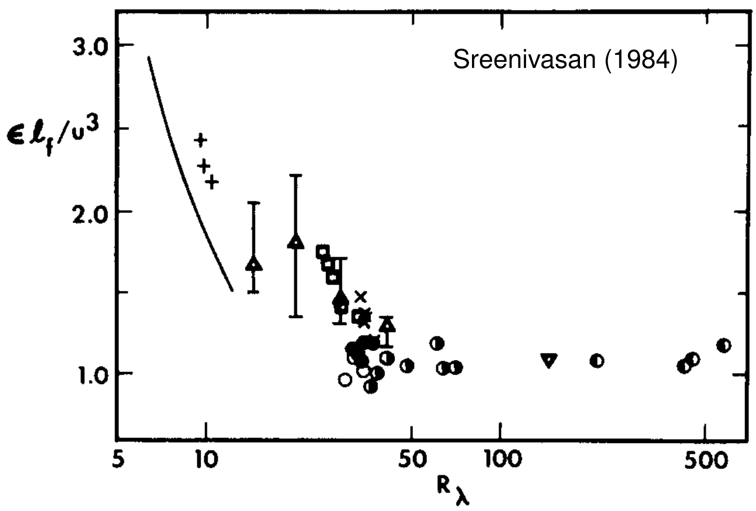
Normalised energy dissipation rate

$$C_\epsilon := \frac{\overbrace{\epsilon}^{\text{Small scale}}}{\underbrace{U^3/L}_{\text{Large scale}}}$$

- $C_\epsilon \to \mathrm{const.}$ as $\mathrm{Re}_\lambda \nearrow$

  $$\mathrm{Re}_\lambda(\boldsymbol{x},t) := \sqrt{\frac{20}{3}}\frac{K(\boldsymbol{x},t)}{\sqrt{\nu\epsilon(\boldsymbol{x},t)}}$$

- Related to LES modelling



“Zeroth law” of turbulence Frisch (1995)

Relation with Kolmogorov’s 1941 theory

Integrating the K41 energy spectrum

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}$$

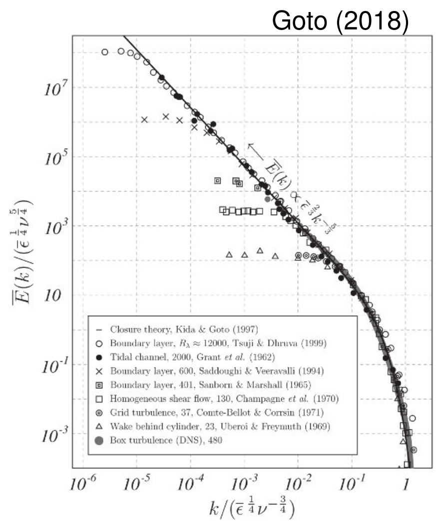
from $L^{-1} \to \infty$ gives

$$\int_{L^{-1}}^{\infty} E_0(k)\,\mathrm{d}k = \int_{L^{-1}}^{\infty} C_K \epsilon^{2/3} k^{-5/3}\,\mathrm{d}k$$

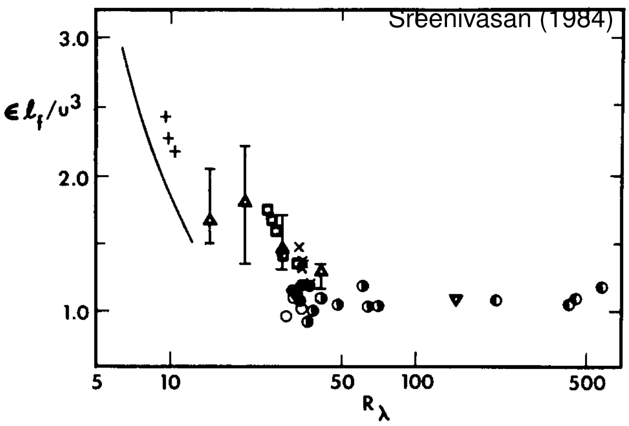
$$K = \frac{3}{2} C_K \epsilon^{2/3} L^{3/2}.$$

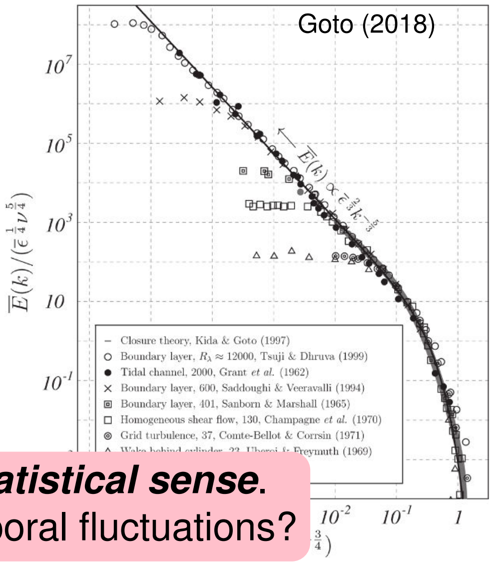
Here, $U = \sqrt{2K/3}$ leads to the dissipation law

$$\epsilon = C_\epsilon \frac{U^3}{L} \quad \text{where} \quad C_\epsilon = (C_K)^{-3/2}.$$



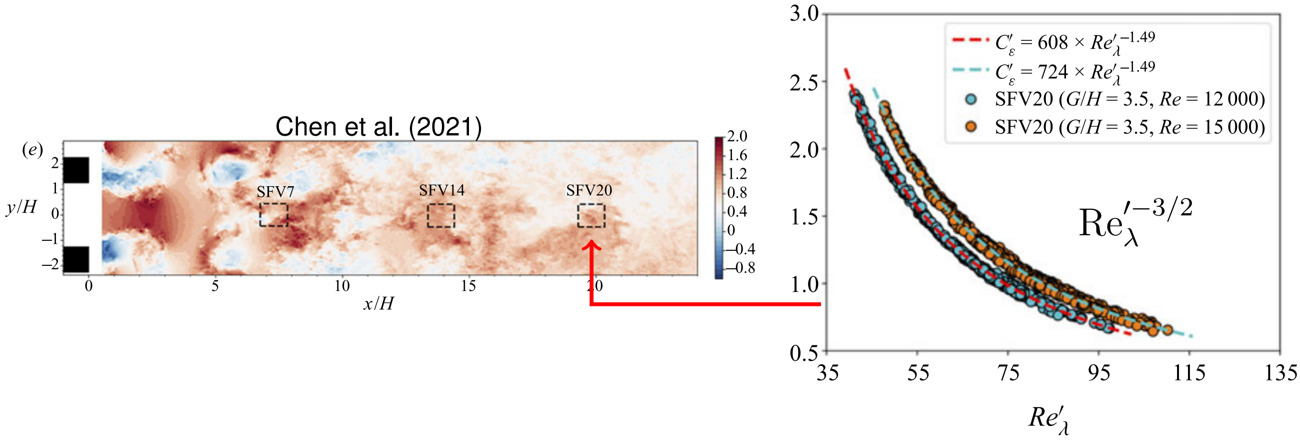
"Zeroth law" of turbulence Frisch (1995)





These scaling laws hold ***in statistical sense***.
What happens with spatio-temporal fluctuations?

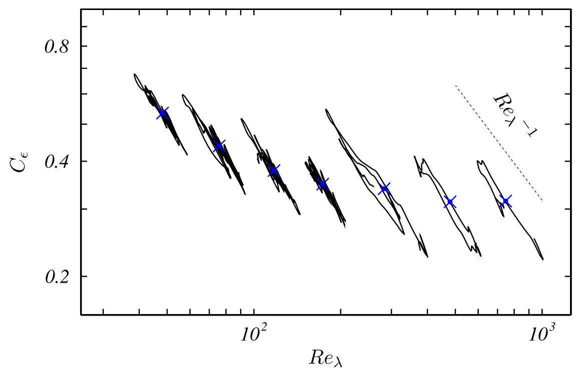
$C_\epsilon(\boldsymbol{x})$–$\mathrm{Re}_\lambda(\boldsymbol{x})$ scaling with spatial fluctuations



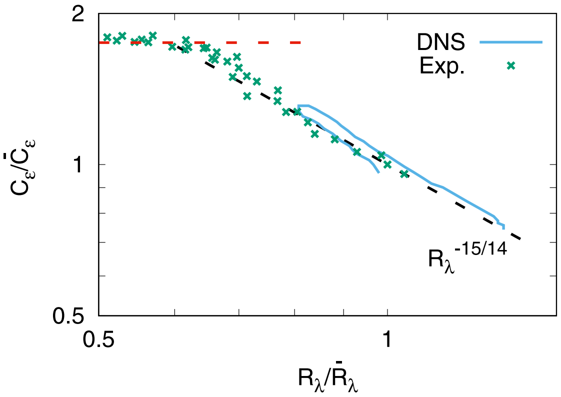
In the wake of cylinders, Chen et al. (2021) observed $C_\epsilon(\boldsymbol{x}) \propto \mathrm{Re}_\lambda(\boldsymbol{x})^{-3/2}$.

$C_\epsilon(t)$–$\mathrm{Re}_\lambda(t)$ scaling with temporal fluctuations

Goto and Vassilicos (2015)



Bos and Rubinstein (2017)



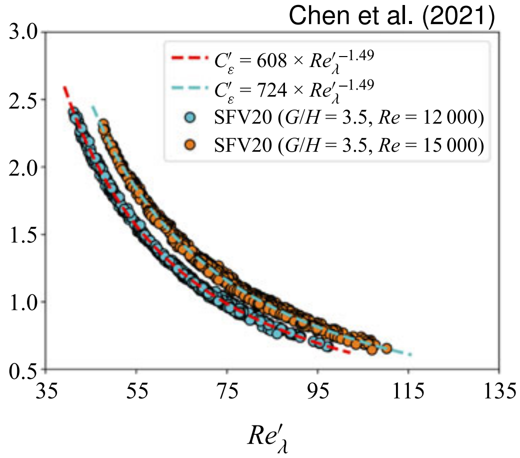
- With steady-forcing DNS in a periodic box, Goto and Vassilicos (2015) reported $C_\epsilon(t) \propto \mathrm{Re}_\lambda(t)^{-1}$.
- Bos and Rubinstein (2017) refined the scaling to $C_\epsilon(t) \propto \mathrm{Re}_\lambda(t)^{-15/14}$.

Fluctuating C_ϵ–Re_λ scaling

Spatial fluctuations

$\rightarrow$ Nicely collapsed data

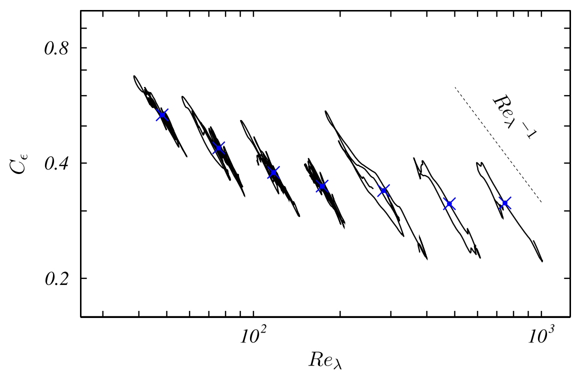
Chen et al. (2021)



Temporal fluctuations

$\rightarrow$ Not perfect collapse

Goto and Vassilicos (2015)



How can we understand these $C_\epsilon(\boldsymbol{x},t)$–$\mathrm{Re}_\lambda(\boldsymbol{x},t)$ scaling laws?

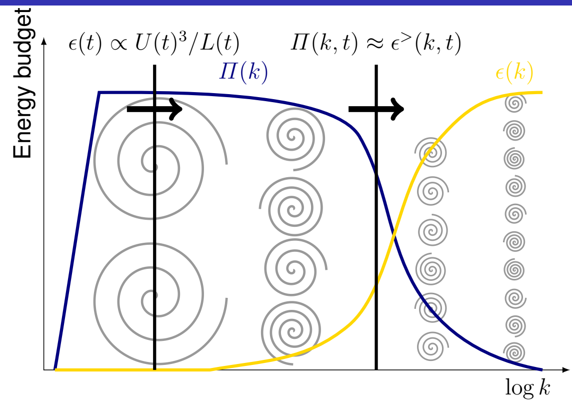
“Equilibrium” and “nonequilibrium” scaling

Local equilibrium hypothesis

- Instantaneous balance between flux and dissipation of energy
$$\Pi(k,t) \approx \epsilon^{>}(k,t).$$
- Valid in “equilibrium” small scales

“Nonequilibrium” large scales

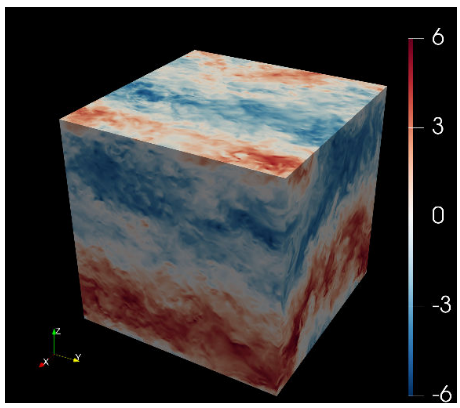
- The hypothesis does not hold in “nonequilibrium” large scales where
$$E(k,\boldsymbol{x},t) = \underbrace{E_0(k)}_{\propto k^{-5/3}} + \underbrace{E_1(k,\boldsymbol{x},t)}_{\propto k^{?}}.$$



- How does $E_1(k,\boldsymbol{x},t)$ scale?
- How does $E_1(k,\boldsymbol{x},t)$ affect $C_\epsilon(\boldsymbol{x},t)$–$\mathrm{Re}_\lambda(\boldsymbol{x},t)$ scaling?

Nonequilibrium scaling of inhomogeneous turbulence

- Assume inhomogeneous mean flow $U(z)$
- Energy equation in (k,z) space:
$$\frac{\mathrm{D}E(k,z)}{\mathrm{D}t} = \underbrace{P(k,z)}_{\text{Production}} - \underbrace{2\nu k^2 E(k,z)}_{\text{Dissipation}} + \underbrace{T(k,z)}_{\text{Transfer}} + \underbrace{D(k,z)}_{\text{Diffusion}}$$



- Assume statistical steady state & inertial range $k_f \ll k \ll k_\eta$
$$-\underbrace{T(k,z)}_{\text{Transfer in } \textbf{\textit{scale}}} = \underbrace{D(k,z)}_{\text{Diffusion in } \textbf{\textit{space}}}$$

Inertial-range balance with fluxes

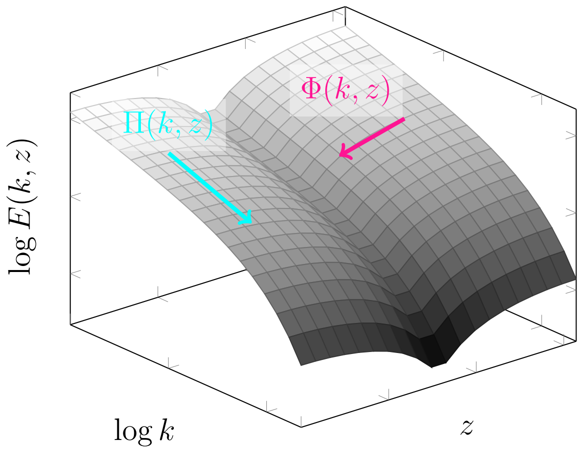
$$-\underbrace{\boxed{\frac{\partial}{\partial k}\Pi(k,z)}}_{T(k,z)} = \underbrace{\boxed{\frac{\partial}{\partial z}\Phi(k,z)}}_{D(k,z)}$$

- Flux in ***scale*** Leith (1967)

$$\boxed{\Pi(k,z)} = -\rho(k,z)\frac{\partial k^{-2}E(k,z)}{\partial k}$$

- Flux in ***space*** Besnard et al. (1996)

$$\boxed{\Phi(k,z)} = -\mu(z)\frac{\partial E(k,z)}{\partial z}$$



Modelling and perturbation analysis

E_0: Equilibrium; E_1: Nonequilibrium

- With turbulent diffusivity in scale $\rho(k,z)$ and space $\mu(z)$,

$$\boxed{\Pi(k,z)} = -\rho(k,z)\frac{\partial k^{-2}E(k,z)}{\partial k} \quad \text{with} \quad \rho(k,z) \sim k^{11/2}E(k,z)^{1/2},$$

$$\boxed{\Phi(k,z)} = -\mu(z)\frac{\partial E(k,z)}{\partial z} \quad \text{with} \quad \mu(z) \sim L^{4/3}\epsilon(z)^{1/3}.$$

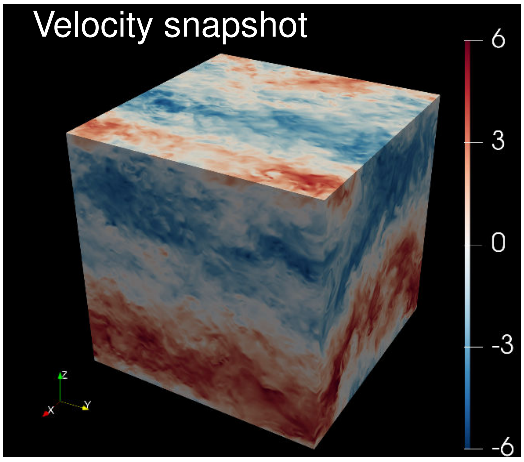
- Perturbation analysis: $E = E_0 + E_1$ where $|E_1| \ll |E_0|$ leads to:

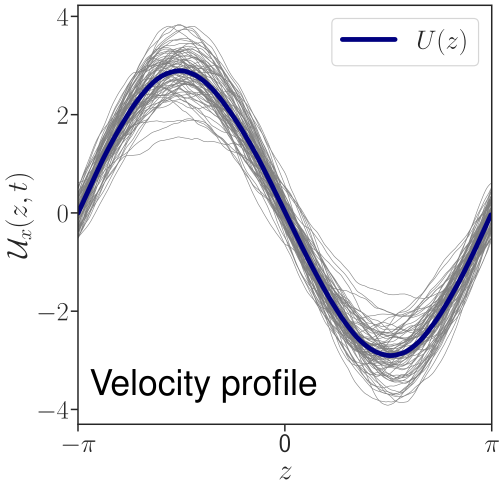
$$E_1(k,z) \sim -\frac{\partial^2\epsilon/\partial z^2\ L^{4/3}}{\epsilon(z)^{1/3}}k^{-7/3}.$$

Large-scale inhomogeneous mean flow induces $E_1(k,z) \propto k^{-7/3}$.

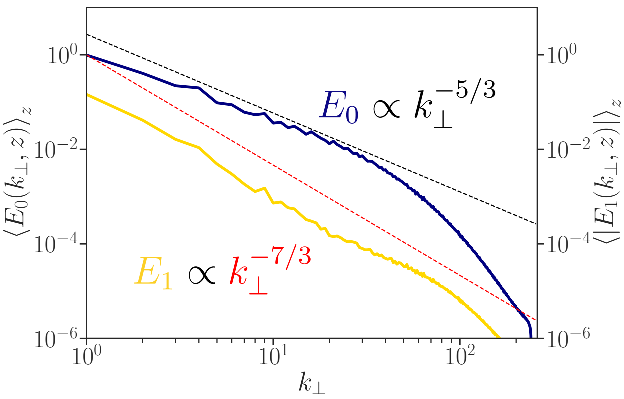
Numerical simulation of inhomogeneous turbulence

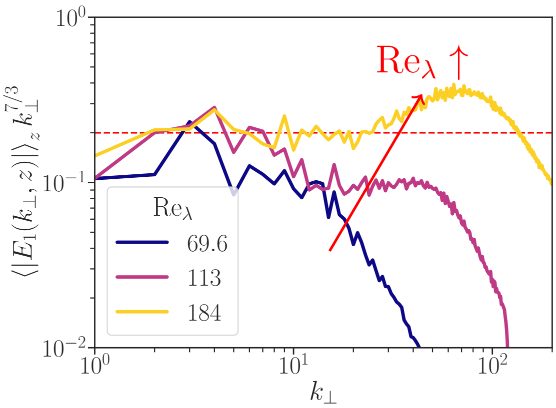
- $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \sin(z)\boldsymbol{e}_x$ → sinusoidal z-profile.
- Inhomogeneous energy spectrum: $E(k_\perp, z)$ where $k_\perp = \sqrt{k_x^2 + k_y^2}$.





Numerical verification of the $E_1(k_\perp) \propto k_\perp^{-7/3}$ scaling





- Nonequilibrium $E_1(k_\perp, z) \propto k_\perp^{-7/3}$ scaling is numerically confirmed.
- Large-scale inhomogeneity → rapidly decaying nonequilibrium scaling.

Ryo Araki and Wouter J. T. Bos (2024). Inertial range scaling of inhomogeneous turbulence. *Journal of Fluid Mechanics* 978, A9

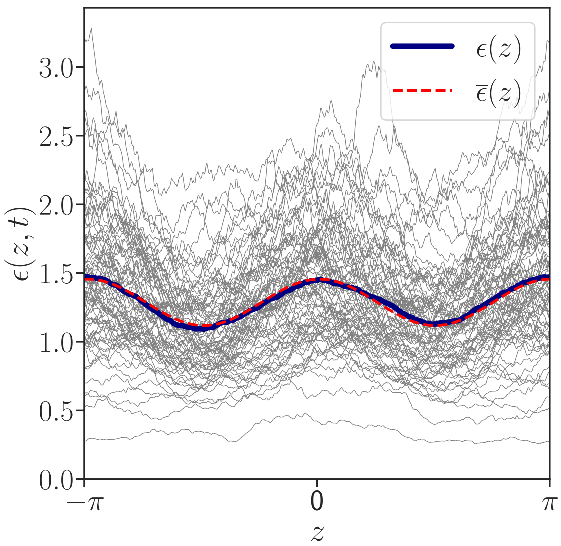
Inhomogeneous $C_\epsilon(z)$–$\mathrm{Re}_\lambda(z)$ scaling

$$C_\epsilon(z) := \frac{\epsilon(z)L}{U(z)^3} \quad \text{and} \quad \mathrm{Re}_\lambda(z) := \frac{20}{3}\frac{K(z)}{\sqrt{\nu\epsilon(z)}}$$

with

$$E_1(k,z) \sim -\frac{\partial^2\epsilon}{\partial z^2} L^{4/3}\epsilon(z)^{-1/3}k^{-7/3}$$

- $K(z) = \int_{L^{-1}}^{\infty} \mathrm{d}k\,[E_0(k,z) + E_1(k,z)]$
- Approx. $\epsilon(z) = \langle\epsilon\rangle\left[1 + \alpha_\epsilon \sin(qz)\right]$
- Employ $L = k_f^{-1}$



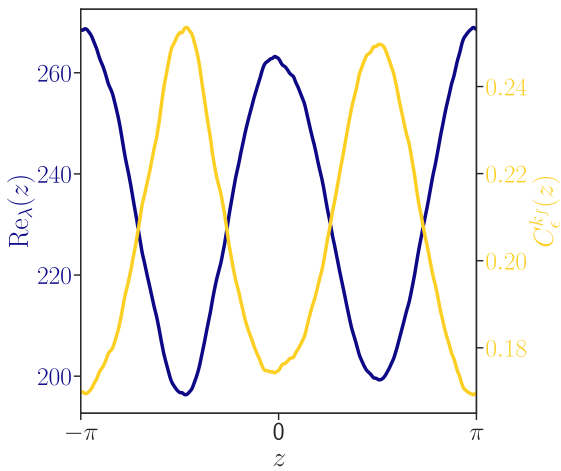
Inhomogeneous $C_\epsilon(z)$–$\mathrm{Re}_\lambda(z)$ scaling

$\epsilon(z) = \langle\epsilon\rangle\,[1 + \alpha_\epsilon \sin(qz)]$ and $X \propto L$

$$C_\epsilon^{k_f}(z) = \frac{\epsilon(z)}{U(z)^3 k_f} \propto 1 - \frac{3}{2}\alpha_\epsilon (qX)^2 \sin(qz),$$

$$\mathrm{Re}_\lambda(z) = \frac{20}{3}\frac{K(z)}{\sqrt{\nu\epsilon(z)}} \propto 1 + \alpha_\epsilon\left[(qX)^2 + 1/6\right]\sin(qz).$$

$$\because E_1 = \mathcal{F}\left(\frac{\partial^2\epsilon}{\partial z^2}\right) \text{ and } \frac{\mathrm{d}^2 \sin(x)}{\mathrm{d}x^2} = -\sin(x)$$



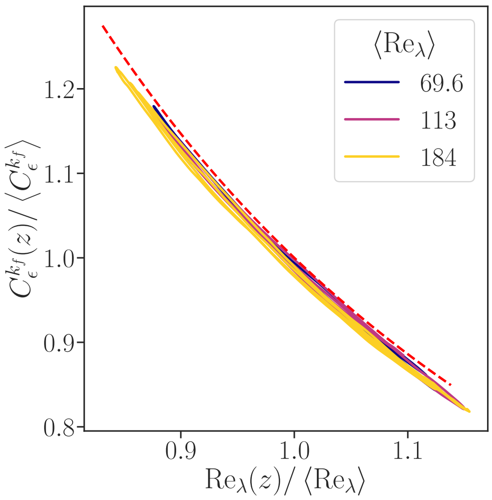
Linear relation: $C_\epsilon^{k_f}(z) \propto -\frac{3}{2}\mathrm{Re}_\lambda(z) \leftarrow \mathrm{Re}_\lambda(z)^{-3/2}$ Chen et al. (2021).

Inhomogeneous $C_\epsilon(z)$–$\mathrm{Re}_\lambda(z)$ scaling

$\epsilon(z) = \langle\epsilon\rangle\,[1+\alpha_\epsilon \sin(qz)]$ and $X \propto L$

$$
\begin{aligned}
C_\epsilon^{k_f}(z) &= \frac{\epsilon(z)}{U(z)^3 k_f} \\
&\propto 1 - \frac{3}{2}\alpha_\epsilon (qX)^2 \sin(qz), \\
\mathrm{Re}_\lambda(z) &= \frac{20}{3}\frac{K(z)}{\sqrt{\nu\epsilon(z)}} \\
&\propto 1 + \alpha_\epsilon\left[(qX)^2 + 1/6\right]\sin(qz).
\end{aligned}
$$

$\because E_1 = \mathcal{F}\left(\frac{\partial^2\epsilon}{\partial z^2}\right)$ and $\frac{\mathrm{d}^2 \sin(x)}{\mathrm{d}x^2} = -\sin(x)$



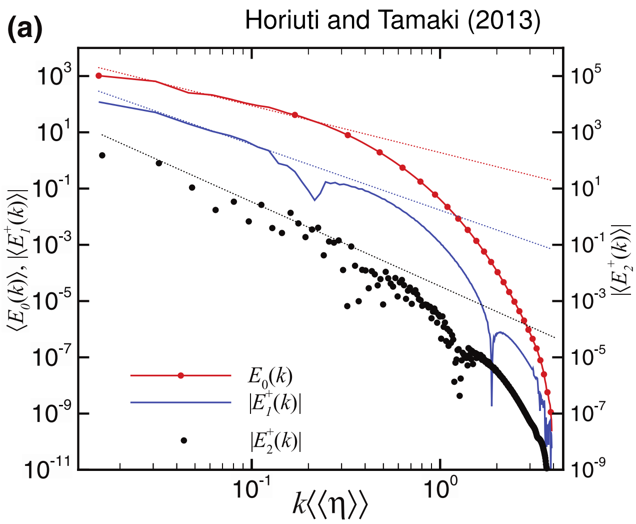
Linear relation: $C_\epsilon^{k_f}(z) \propto -\frac{3}{2}\mathrm{Re}_\lambda(z) \leftarrow \mathrm{Re}_\lambda(z)^{-3/2}$ Chen et al. (2021).

Unsteady scaling for $E_1(k,t)$

According to Yoshizawa (1994),

$$
E(k,t) = \underbrace{C_K \epsilon(t)^{2/3} k^{-5/3}}_{E_0(k,t)} + \underbrace{C_K' \frac{\mathrm{d}\epsilon/\mathrm{d}t}{\epsilon(t)^{2/3}} k^{-7/3}}_{E_1(k,t)}.
$$

Large-scale unsteady flow induces $E_1(k,t) \propto k^{-7/3}$.



Unsteady $C_\epsilon(t)$–$\mathrm{Re}_\lambda(t)$ scaling

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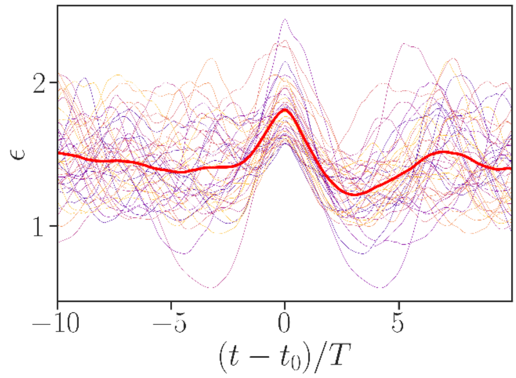
with

$$E_1(k,t) \sim \frac{\mathrm{d}\epsilon}{\mathrm{d}t}\epsilon(t)^{-2/3}k^{-7/3}$$

- $K(t) = \int_{L^{-1}}^{\infty} \mathrm{d}k\,[E_0(k,t) + E_1(k,t)]$
- Approx. $\epsilon(t) = \langle\epsilon\rangle\left[1 + \alpha_\epsilon \sin(\omega t)\right]$
- Employ $L = k_f^{-1}$



Unsteady $C_\epsilon(t)$–$\mathrm{Re}_\lambda(t)$ scaling

$\epsilon(t) = \langle\epsilon\rangle\,[1 + \alpha_\epsilon \sin(\omega t)]$ and $\mathcal{T} \propto (L^2/\epsilon)^{1/3}$

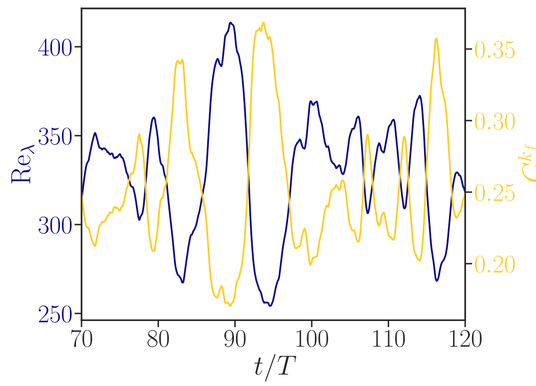
$$C_\epsilon^{k_f}(t) = \frac{\epsilon(t)}{U(t)^3 k_f}$$

$$\propto 1 - \frac{3}{2}\alpha_\epsilon\omega\mathcal{T}\cos(\omega t),$$

$$\mathrm{Re}_\lambda(t) = \frac{20}{3}\frac{K(t)}{\sqrt{\nu\epsilon(t)}}$$

$$\propto 1 + \alpha_\epsilon\left[\omega\mathcal{T}\cos(\omega t) + \frac{1}{6}\sin(\omega t)\right].$$

$\because E_1 = \mathcal{F}\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\right)$ and $\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \cos(x)$



Elliptic relation between $C_\epsilon^{k_f}(t) \propto -\frac{3}{2}\mathrm{Re}_\lambda(t) \leftarrow \mathrm{Re}_\lambda(t)^{-3/2}$.

Unsteady $C_\epsilon(t)$–$\mathrm{Re}_\lambda(t)$ scaling

$\epsilon(t) = \langle\epsilon\rangle\,[1 + \alpha_\epsilon \sin(\omega t)]$ and $\mathcal{T} \propto (L^2/\epsilon)^{1/3}$

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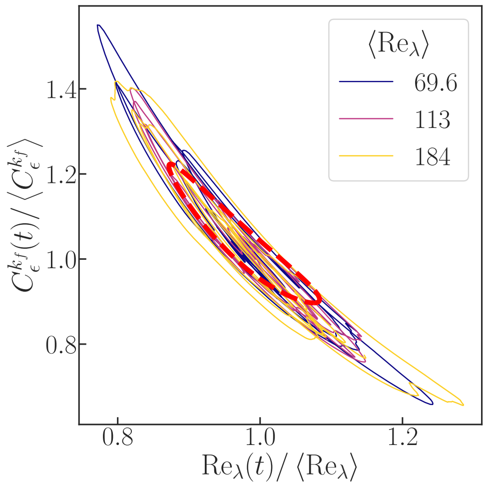
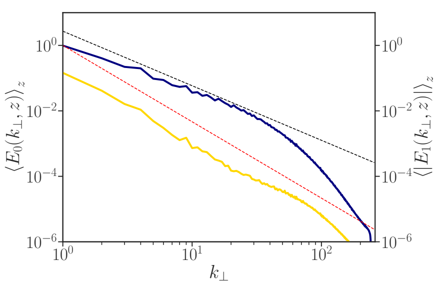


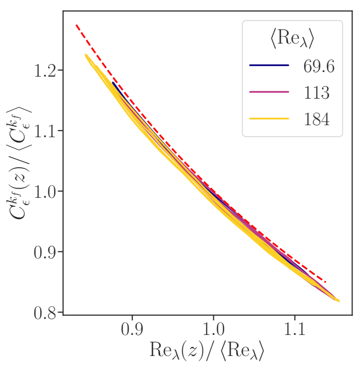
Elliptic relation between $C_\epsilon^{k_f}(t) \propto -\frac{3}{2}\mathrm{Re}_\lambda(t) \leftarrow \mathrm{Re}_\lambda(t)^{-3/2}$.

Summary

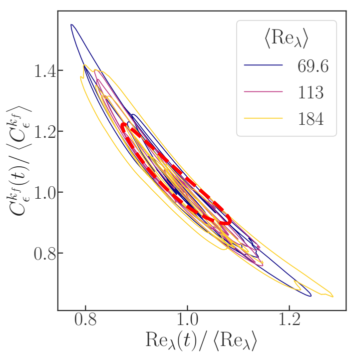
- Understand the nonequilibrium scaling of normalised dissipation rate for inhomogeneous $C_\epsilon(\boldsymbol{x})$ and unsteady $C_\epsilon(t)$ flows.
- Method: Linear perturbation analysis of the nonequilibrium energy spectrum $E_1(k, \boldsymbol{x}, t)$.



(a) $E_0(k,z)$ and $E_1(k,z)$



(b) Linear scaling of $C_\epsilon^{k_f}(z)$



(c) Elliptic scaling of $C_\epsilon^{k_f}(t)$

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