Oral presentation | Higher order methods

Higher order methods-IV

Tue. Jul 16, 2024 4:30 PM - 6:30 PM Room C

[6-C-03] An unstructured high-order compact gas-kinetic scheme in arbitrary Lagrangian-Eulerian formulation

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Keywords: arbitrary Lagrangian-Eulerian (ALE), geometric conservation law (GCL), compact gas-kinetic scheme, multi-stage multi-derivative

An unstructured high-order compact gas-kinetic scheme in arbitrary Lagrangian-Eulerian formulation

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Outline

1. Motivation
2. GKS in ALE Formulation
3. High order method
4. Numerical experiments
5. Conclusion

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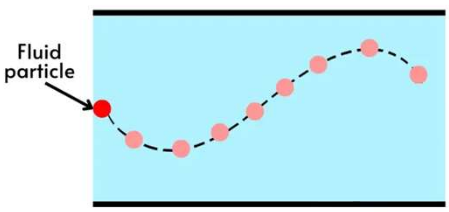
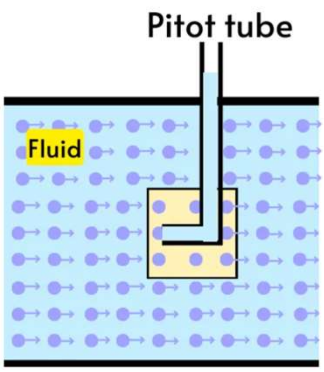
Motivation/Lagrangian and Eulerian Discerption

- **Lagrangian method**



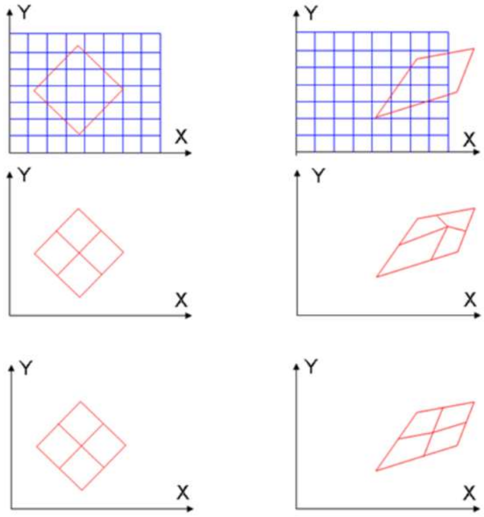
The observer follows a fluid particle/parcel to observe changes in its properties.

- **Eulerian method**



The fluid is observed through a fixed point or control volume.

Motivation/Arbitrary Lagrangian-Eulerian



Eulerian mesh

Lagrangian mesh

ALE mesh

Initial state

Deformed configuration

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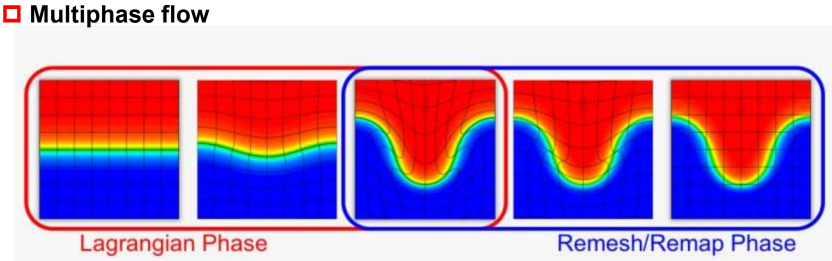
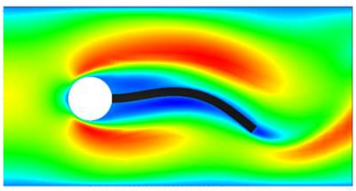
Motivation/Application of Arbitrary Lagrangian-Eulerian

- Multiphase flow



- Fluid-structure interaction



- Blast

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Outline

GKS in ALE Formulation/Govern Eqn and GCL

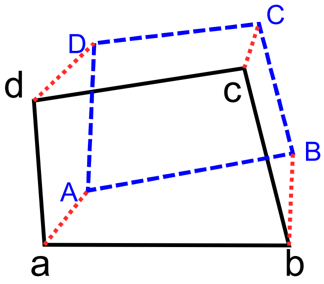
- **Governing equation on moving mesh**

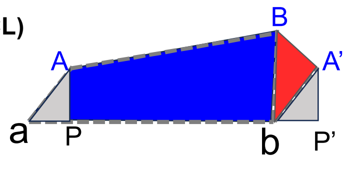
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_i(t)} \mathbf{W}\mathrm{d}\Omega + \int_{\partial\Omega_i} (\mathbf{F}(\mathbf{W}) - \mathbf{W}\mathbf{U})\cdot \mathbf{n}\mathrm{dS}=0$$

If $\mathbf{U}=\mathbf{V}$, the governing equation becomes Lagrangrian form;

If $\mathbf{U}=\mathbf{0}$, the governing equation becomes Eulerian form;

- **Geometrical Conservation Law (GCL)**





$$S_{abBA} = S_{APP'A'B} - S_{A'Bb}$$

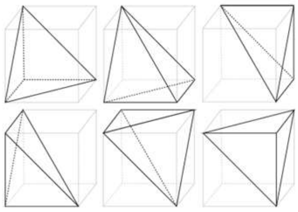
$$S_{APP'A'B} = \frac{1}{2}|PP'|(U_a + U_b)\Delta t = \frac{1}{2}|ab|(U_a + U_b)\Delta t$$

where U_a and U_b are the velocity of point a and b along the normal direction of line ab

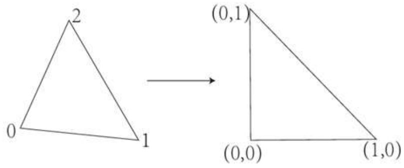
GKS in ALE Formulation/Finite Volume Discretion

Finite volume method

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \mathbf{W}\mathrm{d}\Omega + \sum_{p_l=1}^{N_l}\int_{\Sigma_{p_l}} (\mathbf{F}(\mathbf{W}) - \mathbf{W}\mathbf{U})\cdot \mathbf{n}\mathrm{d}S = 0,$$



Coordinates translation



$$\mathbf{x}(\xi,\eta) = \mathbf{x}_0 + \xi(\mathbf{x}_1 - \mathbf{x}_0) + \eta(\mathbf{x}_2 - \mathbf{x}_0),$$

$$\mathbf{n} = \mathbf{x}_{,\xi} \times \mathbf{x}_{,\eta} = \frac{\partial \mathbf{x}}{\partial \xi} \times \frac{\partial \mathbf{x}}{\partial \eta}$$

$$\mathbf{x} = \mathbf{x}^0 + \mathbf{U}t$$

Flux calculation

$$\int_{\Sigma_{p_l}} (\mathbf{F}(\mathbf{W}) - \mathbf{W}\mathbf{U})\cdot \mathbf{n}\mathrm{d}S \approx \mathcal{L}_{p_l}(\mathbf{W},\Omega) + \mathbb{V}_{p_l}(t)$$

$$\mathcal{L}_{p_l}(\mathbf{W},\Omega) = \sum_{k=1}^{3}\omega_k \left[\mathbf{F}(\mathbf{W}_{p_l,k}) - \mathbf{W}_{p_l,k}\mathbf{U}_{p_l,k}\right]\cdot(\mathbf{x}^0_{,\xi} \times \mathbf{x}^0_{,\eta})_{p_l},$$

$$\mathbb{V}_{p_l}(t) = \sum_{k=1}^{3}\omega_k \mathbf{W}_{p_l,k}\mathbf{U}_{p_l,k}\cdot\left[t(\mathbf{U}^0_{,\xi} \times \mathbf{x}^0_{,\eta}) + t(\mathbf{x}^0_{,\xi} \times \mathbf{U}^0_{,\eta}) + t^2(\mathbf{U}^0_{,\xi} \times \mathbf{U}^0_{,\eta})\right]_{p_l}.$$

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GKS in ALE Formulation/BGK eqn in ALE Formulation

BGK equation on moving reference

$$\frac{\partial f}{\partial t} + \mathbf{w}\cdot\nabla_x f = \frac{g - f}{\tau}$$

where $\mathbf{w} = \mathbf{v} - \mathbf{U}.$

Flux with considering mesh moving

$$\mathbb{F}(\mathbf{W}) = (\mathbf{F}(\mathbf{W}) - \mathbf{W}\mathbf{U})\cdot(\mathbf{x}^0_{,\xi} \times \mathbf{x}^0_{,\eta}) = \int f(\mathbf{x},t,\mathbf{w},\xi)\mathbf{w}\cdot\mathbf{n}\boldsymbol{\Psi}\mathrm{d}\mathbf{v}\mathrm{d}\Xi$$

where $\boldsymbol{\Psi} = (1, v_1, v_2, v_3, \frac{1}{2}(v_1^2 + v_2^2 + v_3^2 + \xi^2))^T$

$$\mathbb{F}(\mathbf{W})' = \int f(\mathbf{x},t,\mathbf{w},\xi)\mathbf{w}\cdot\mathbf{n}\boldsymbol{\Psi}'\mathrm{d}\mathbf{v}\mathrm{d}\Xi,$$

where $\boldsymbol{\Psi}' = (1, w_1, w_2, w_3, \frac{1}{2}(w_1^2 + w_2^2 + w_3^2 + \xi^2))^T$

$$\begin{cases} \mathbb{F}_\rho = \mathbb{F}'_\rho \\ \mathbb{F}_{\rho V_1} = \mathbb{F}'_{\rho W_1} + U_1\mathbb{F}'_\rho \\ \mathbb{F}_{\rho V_2} = \mathbb{F}'_{\rho W_2} + U_2\mathbb{F}'_\rho \\ \mathbb{F}_{\rho V_3} = \mathbb{F}'_{\rho W_3} + U_3\mathbb{F}'_\rho \\ \mathbb{F}_{\rho E} = \mathbb{F}'_{\rho E} + U_1\mathbb{F}'_{\rho W_1} + U_2\mathbb{F}'_{\rho W_2} + U_3\mathbb{F}'_{\rho W_3} + \frac{1}{2}(U_1^2 + U_2^2 + U_3^2)\mathbb{F}'_\rho \end{cases}$$

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GKS in ALE Formulation/2nd-order Gas-Kinetic Flux

- **Analytical solution of BGK equation**

$$f(\mathbf{x},t,\mathbf{w},\xi)=\frac{1}{\tau}\int_0^t g(\mathbf{x}',t',\mathbf{w},\xi)e^{-(t-t')/\tau}\mathrm{d}t'+e^{t/\tau}f_0(\mathbf{x}_0,\mathbf{w})$$

- **The initial distribution function**

$$f_0=\begin{cases} g^l\left[1-\left(\mathbf{a}^l\cdot\mathbf{w}\right)t-\tau\left(A^l+\mathbf{a}^l\cdot\mathbf{w}\right)\right], & x_1<0,\\ g^r\left[1-\left(\mathbf{a}^r\cdot\mathbf{w}\right)t-\tau\left(A^r+\mathbf{a}^r\cdot\mathbf{w}\right)\right], & x_1\geq 0,\end{cases}$$

- **The equilibrium distribution function**

$$g(\mathbf{x}',t',\mathbf{w},\xi)=\bar{g}(\mathbf{x},0,\mathbf{w},\xi)\left\{1-\bar{\mathbf{a}}\cdot\mathbf{w}(t-t')+At'\right\},$$

with denoting

$$\mathbf{a}=(a_1,a_2,a_3)=\nabla_x g/g, A=g_t/g.$$

- **Time-accurate conservative value at cell interface**

$$\mathbf{W}_{p_i,k}(t^{n+1})=\int\Psi f(\mathbf{x}_{p_i,k},t^{n+1},\mathbf{w},\xi)\mathrm{d}\mathbf{w}\mathrm{d}\Xi$$

According to Gauss's theorem, the cell averages of space partial derivatives can be obtained.

GKS in ALE Formulation/Mesh moving method

- **Radial Basic Function (RBF) Method**

First we know the position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ of boundary points,

Define an interpolation function

$$f(\mathbf{x})=\sum_i\lambda_i\psi(|\mathbf{x}-\mathbf{x_i}|/R),\psi(\xi)=(1-\xi)^4(1+4\xi).$$

By solving equation

$$f(\mathbf{x}_j)=\mathbf{v}_j,$$

we can get λ_i . Then we can obtain the velocity of inner points.

- **Lagrangian Velocity Method**

$$\mathbf{U}_p=\mathbb{M}_p^{-1}\sum\nolimits_{c\in C(p)}\sum\nolimits_{f\in F_p(c)}[S_f p_c\mathbf{N}_f^c+\mathbb{M}_{pcf}\mathbf{V}_c],\mathbb{M}_p=\sum\nolimits_{c\in C(p)}\sum\nolimits_{f\in F_p(c)}\mathbb{M}_{pcf}$$

$$\mathbb{M}_{pcf}=S_f\rho_c a_c(\mathbf{N}_f^c\otimes\mathbf{N}_f^c),$$

$C(p)$ is the set of cells c that share the common vertex p

$F_p(c)$ is the set of faces of cell c that share the common vertex p

Outline

High order method/Two-stage temporal discretization

- **The two-stage fourth-order (S2O4) temporal discretization**

$$(\Omega_i \mathbf{W}_i)^{*} = (\Omega_i \mathbf{W}_i)^{n} + \frac{1}{2}\Delta t \mathcal{L}\left(\mathbf{W}_i^n, \Omega^n\right) + \frac{1}{8}\Delta t^2 \frac{\partial}{\partial t}\mathcal{L}\left(\mathbf{W}_i^n, \Omega^n\right) + \int_0^{1/2\Delta t} \mathbb{V}(t)\mathrm{d}t,$$

$$(\Omega_i \mathbf{W}_i)^{(n+1)} = (\Omega_i \mathbf{W}_i)^{n} + \Delta t \mathcal{L}\left(\mathbf{W}_i^n, \Omega^n\right) + \frac{1}{6}\Delta t^2 \left(\frac{\partial}{\partial t}\mathcal{L}\left(\mathbf{W}_i^n, \Omega^n\right) + 2\frac{\partial}{\partial t}\mathcal{L}\left(\mathbf{W}_i^{*}, \Omega^{*}\right)\right) + \int_0^{\Delta t} \mathbb{V}(t)\mathrm{d}t,$$

With a uniform flow field, the time evolution becomes

$$(\Omega_i)^{*} = (\Omega_i)^{n} + \frac{1}{2}\Delta t \sum_{p_t=1}^{Nt}\sum_{k=1}^{3} \omega_k \mathbf{U}_{p_t,k} \cdot (\mathbf{x}_{,\xi}^0 \times \mathbf{x}_{,\eta}^0)_{p_t}$$
$$+\frac{1}{4}\Delta t^2 \sum_{p_t=1}^{Nt}\sum_{k=1}^{3} \omega_k \mathbf{U}_{p_t,k} \cdot \left[\frac{1}{2}(\mathbf{U}_{,\xi}^0 \times \mathbf{x}_{,\eta}^0) + \frac{1}{2}(\mathbf{x}_{,\xi}^0 \times \mathbf{U}_{,\eta}^0) + \frac{1}{6}\Delta t(\mathbf{U}_{,\xi}^0 \times \mathbf{U}_{,\eta}^0)\right]_{p_t},$$

$$(\Omega_i)^{n+1} = (\Omega_i)^{n} + \Delta t \sum_{p_t=1}^{Nt}\sum_{k=1}^{3} \omega_k \mathbf{U}_{p_t,k} \cdot (\mathbf{x}_{,\xi}^0 \times \mathbf{x}_{,\eta}^0)_{p_t}$$
$$+\Delta t^2 \sum_{p_t=1}^{Nt}\sum_{k=1}^{3} \omega_k \mathbf{U}_{p_t,k} \cdot \left[\frac{1}{2}(\mathbf{U}_{,\xi}^0 \times \mathbf{x}_{,\eta}^0) + \frac{1}{2}(\mathbf{x}_{,\xi}^0 \times \mathbf{U}_{,\eta}^0) + \frac{1}{3}\Delta t(\mathbf{U}_{,\xi}^0 \times \mathbf{U}_{,\eta}^0)\right]_{p_t}$$

which automatically satisfies the Geometrical Conservation Law (GCL).

- **The two-step evolution of gas distribution function**

$$f^{*} = f^{n} + \frac{1}{2}\Delta t f_t^{n},$$
$$f^{n+1} = f^{n} + \Delta t f_t^{*}.$$

High order method/Third-order compact reconstruction

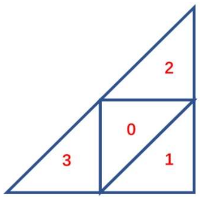
- **3rd-order compact reconstruction for large stencil**

  The cell averages over both self cell and neighbor cells need to be exactly satisfied

$$\iiint_{\Omega_0} p^2 \mathrm{d}V = \bar{Q}_0 |\Omega_0|, \iiint_{\Omega_m} p^2 \mathrm{d}V = \bar{Q}_m |\Omega_m|,$$

  The slop of neighbor cells need to be satisfied in a least-square sense

$$\iiint_{\Omega_m} \frac{\partial}{\partial x_i} p^2 \mathrm{d}V = \alpha_m \left(\bar{Q}_{x_i}\right)_m |\Omega_m|, i = 1,2,3$$



- **Green-Gauss reconstruction for the sub stencil**

$$p^1 = \bar{Q} + \alpha_0 \frac{1}{|\Omega_0|} \mathbf{x} \cdot \sum_{m=1}^{N_f} \frac{\bar{Q}_m + \bar{Q}_0}{2} S_m \mathbf{n}_m.$$

- **Nonlinear process**

$$P_2 = \frac{1}{\gamma_2} p^2 - \frac{\gamma_1}{\gamma_2} p^1, P_1 = p^1$$

$$p^2 = \gamma_1 P_1 + \gamma_2 P_2, (\gamma_1 = \gamma_2 = 0.5)$$

WENO ⟹ $R(\mathbf{x}) = \omega_2 P_2 + \omega_1 P_1.$

- **Gradient compression Factor**

$$\alpha_i = \prod_{p_i=1}^{n_i} \prod_{k=0}^{M_p} \alpha_{p_i,k} = \prod_{p_i=1}^{n_i} \prod_{k=0}^{M_p} \frac{1}{1 + A_{p_i,k}^{\;2}},$$

$$A = \frac{|p^l - p^r|}{p^l} + \frac{|p^l - p^r|}{p^r} + (\mathrm{Ma}_n^l - \mathrm{Ma}_n^r)^2 + (\mathrm{Ma}_t^l - \mathrm{Ma}_t^r)^2$$

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Outline

1. Motivation
2. GKS in ALE Formulation
3. High order method
4. **Numerical experiments**
5. Conclusion

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Numerical experiments/Accuracy test

- **Time-accuracy solution**

$$\rho(x,y,z)=1+0.2\sin(2\pi x_r)\sin(2\pi y_r)\sin(2\pi z_r),$$
$$p(x,y,z)=1, V_1=V_2=V_3=1,$$

where

$$(x_r,y_r,z_r)=(x,y,z)-(1,1,1)t$$

- **Mesh moving function**

Type 1

$$\begin{cases} x=x_0+0.1\sin(2\pi t)\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0) \\ y=y_0+0.1\sin(2\pi t)\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0) \\ z=z_0+0.1\sin(2\pi t)\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0) \end{cases}$$

Type 2

$$\begin{cases} x=x_0+0.05\sin(2\pi t)(\sin(2\pi x_0)+\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0)) \\ y=y_0+0.05\sin(2\pi t)(\sin(2\pi y_0)+\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0)) \\ z=z_0+0.05\sin(2\pi t)(\sin(2\pi z_0)+\sin(2\pi x_0)\sin(2\pi y_0)\sin(2\pi z_0)) \end{cases}$$

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Numerical experiments/Accuracy test

	L_1 error	Order	L_2 error	Order	L_∞ error	Order
Stationary						
10^3	1.39E-02		1.81E-02		5.16E-02	
20^3	2.07E-03	2.75	2.79E-03	2.70	7.92E-03	2.70
40^3	2.68E-04	2.95	3.61E-04	2.95	1.05E-03	2.92
80^3	3.27E-05	3.03	4.55E-05	2.99	1.33E-04	2.98
Type 1						
10^3	1.53E-02		1.98E-02		5.66E-02	
20^3	3.19E-03	2.27	3.79E-03	2.38	9.78E-03	2.53
40^3	4.79E-04	2.73	5.56E-04	2.77	1.31E-03	2.90
80^3	6.25E-05	2.94	7.22E-05	2.94	1.66E-04	2.98
Type 2						
10^3	1.37E-02		1.80E-02		5.15E-02	
20^3	2.05E-03	2.75	2.77E-03	2.70	8.01E-03	2.68
40^3	2.64E-04	2.96	3.59E-04	2.95	1.06E-03	2.92
80^3	3.32E-05	2.99	4.52E-05	2.99	1.34E-04	2.99

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Numerical experiments/Geometrical Conservation Law

- **Geometrical Conservation Law (GCL)**

$$\rho(x,y,z)=1,$$
$$p(x,y,z)=1, V_1=V_2=V_3=1,$$

	L_1 error	L_2 error	L_∞ error
Type 1			
10^3	1.44E-15	1.90E-15	8.22E-15
20^3	2.78E-15	3.69E-15	2.02E-14
40^3	5.91E-15	8.05E-15	5.73E-14
80^3	1.18E-14	1.63E-14	1.33E-13
Type 2			
10^3	1.65E-15	2.16E-15	8.22E-15
20^3	3.46E-15	4.60E-15	2.41E-14
40^3	7.18E-15	9.71E-15	5.90E-14
80^3	1.41E-14	1.96E-14	1.70E-13

Numerical experiments/Flow in a cylinder

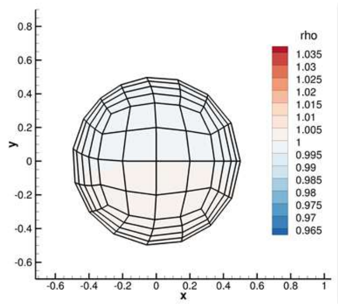
- **Composite of four motions**

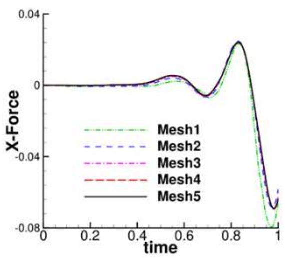
  Translation

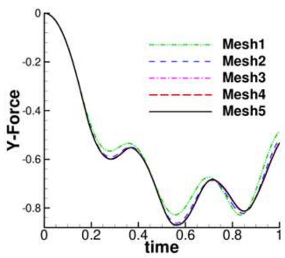
  Rotation

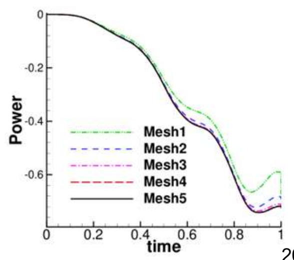
  Const-volume deformation

  Non-unit geometry mapping Jacobian









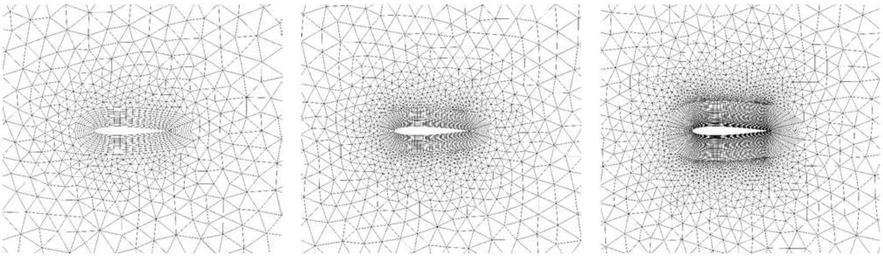
Numerical experiments/Heavingpitching airfoil

- Flow condition

$$\mathrm{Ma}=0.2, \mathrm{Re}=1000, \mathrm{Pr}=0.72$$

$$\rho_\infty=1.0, T_\infty=25, U_\infty=1$$

- Meshes



Mesh 1 Mesh 2 Mesh 3

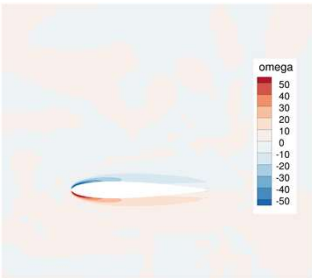
Numerical experiments/Heaving-pitching airfoil

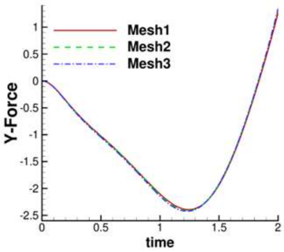
- Translation

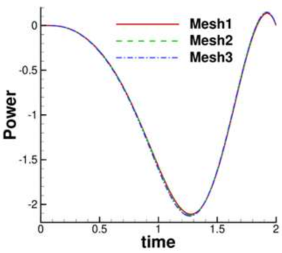
$$\Delta h(t)=t^3(8-3t)/16$$

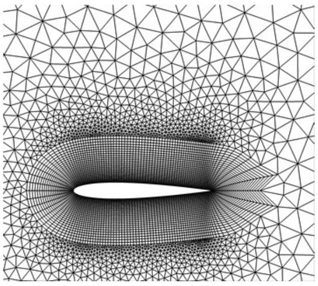
Motion of boundary is determined;

Motion of inner nodes is obtained by radial basic function method









Numerical experiments/Heaving-pitching airfoil

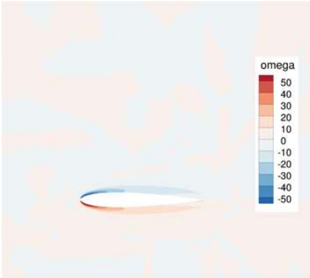
- **Translation + rotation**

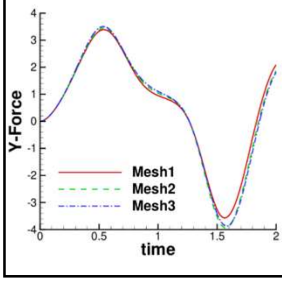
$$\Delta h(t) = t^3(8-3t)/16$$

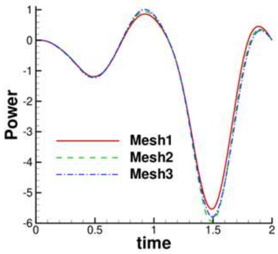
$$\Delta\theta(t) = \frac{80\pi}{180}(-t^6+6t^5-12t^4+8t^3).$$

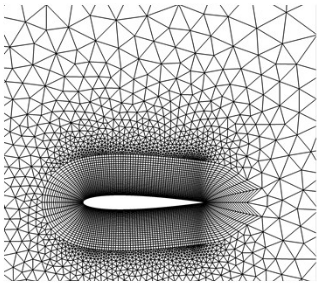
Motion of boundary is determined;

Motion of inner nodes is obtained by radial basic function method









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Numerical experiments/2D Noh Problem

- **Initial condition**

$$\rho = 1$$

$$e = \frac{p}{\gamma-1} = 1\times10^{-4}, \gamma = 5/3$$

$$\mathbf{V} = (-x/r, -y/r)$$

- **Boundary condition**

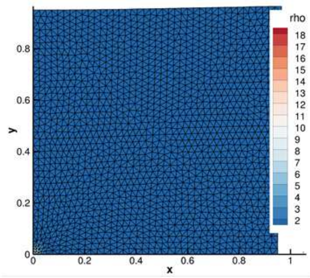
Symmetric boundary condition for $x=0$ and $y=0$

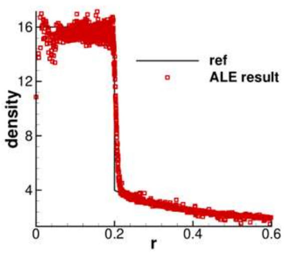
Non-reflective boundary condition for other boundaries

- **Exact solution**

$$\rho = \begin{cases} 16, & r<0.2 \\ 1+t/r, & r>0.2 \end{cases}, \text{ when } t=0.6$$

- **Lagrangian velocity method are used**





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Numerical experiments/Sedov Problem

- **Initial condition**

$\rho = 1$

$$p = \begin{cases} (\gamma-1)\epsilon_0 / V_0, & \text{cell containing the origin} \\ 1\times 10^{-6}, & \text{others} \end{cases}$$

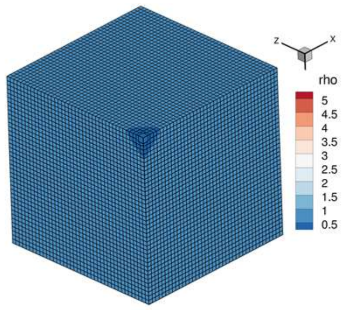
$\mathbf{V} = 0,$

$\epsilon_0 = 0.106384$

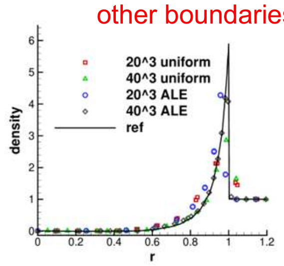
- **Boundary condition**

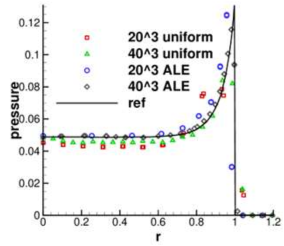
Symmetric boundary condition for $x=0,\ z=0$ and $y=0$

Non-reflective boundary condition for other boundaries



- **Lagrangian velocity method are used**





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Numerical experiments/Saltzmann Problem

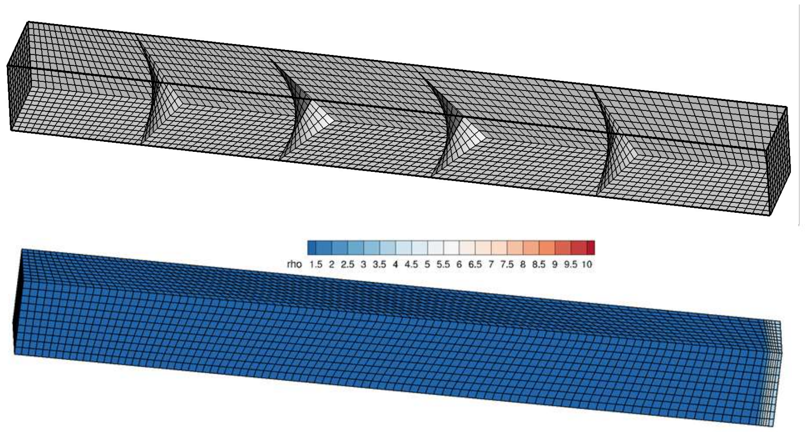
- **Initial condition**

$\rho = 1, e = 10^{-4}, \gamma = 5/3$

- **Lagrangian velocity method are used**

- **Boundary condition**

The wall on the left-hand side moves at a constant speed of 1 from left to right, acting as a piston. The other boundaries act as inviscid reflecting walls.



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Outline

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Conclusion

- A third-order compact gas-kinetic scheme is developed in ALE formulation for three-dimensional unstructured mesh.

- With the help of radial basic function interpolation, the method can handle moving boundary problems.

- For blast problems with strong discontinuities, the scheme retains robustness, and the Lagrangian nodal solver is adapted to track material interfaces.

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Thanks for your attention!

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