Oral presentation | Higher order methods

Higher order methods-III

Tue. Jul 16, 2024 2:00 PM - 4:00 PM Room C

[5-C-04] A robustness-enhanced reconstruction based on discontinuity feedback factor for high-order finite volume scheme

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Keywords: Robustness, High-order finite volume scheme, Discontinuity feedback factor

A robustness-enhanced reconstruction based on discontinuity feedback factor for high-order finite volume scheme

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Why high-order

The success of second-order method

- In the 1990s, CAE software utilizing the second-order method and Reynolds-average Navier-Stokes equations had achieved lots of successful application in aircraft design.

ANSYS FLUENT

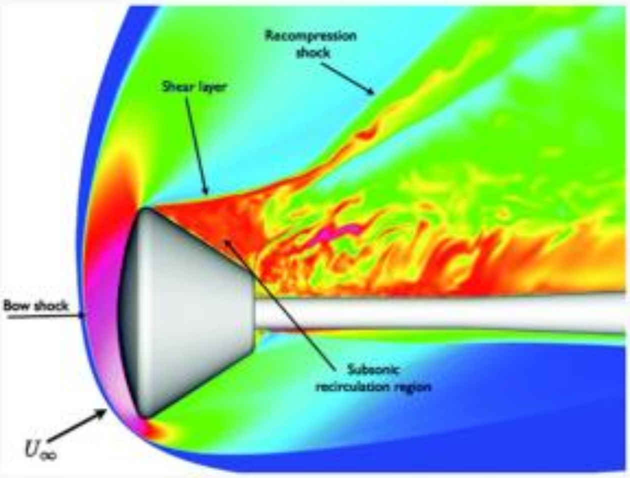
Reconstruction a linear function distribution within an element

$$W(\mathbf{x}) = W_0 + W_{x_i}(\mathbf{x})$$

[1] Witherden, Freddie D., and Antony Jameson, 23rd AIAA Computational Fluid Dynamics Conference. 2017. ,
[2] Goldhammer, M. I., CEAS/KATnet Conference On Key Aerodynamic Technologies, Bremen, 2005.

Why high-order

In hypersonic case



- Temperature contour
- Large temperature gradient (and thermal boundary layer) in the nose of re-entry vehicle
- Subsonic recirculation region (unsteady-turbulent) in the tail

High-order method is more efficient for high-order term and turbulence

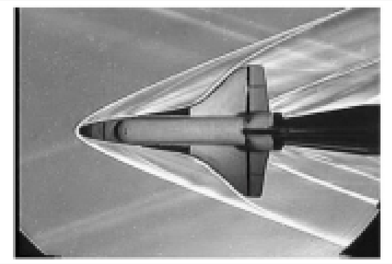
[1]Brock J M, Subbareddy P K, Candler G V. AIAA journal, 2015.

How high-order

Challenge for hypersonic applications

In comparison with second-order method

- Lack of robustness facing strong shock/rarefaction wave.
- Incompatible data structure with second-order method, hard to program
- Less Computational cost



Hypersonic vehicle:

- High Mach
- High Re
- Complicated geometry

Needed:

- Robustness
- Efficiency
- Portability

GAS-KINETIC SCHEME

BGK kinetic model

$$\underbrace{f_t + u \cdot \nabla f}_{\text{Free transport}} = \underbrace{\frac{g - f}{\tau}}_{\text{Collision}} \xrightarrow{\text{1}^{\text{st}}\text{–order C-E expansion}} \text{N-S equation: } \mathbf{W}_t + \nabla \cdot \mathbf{F}(\mathbf{W}, \nabla \mathbf{W}) = 0$$

gas distribution function gives

$$f = f(\mathbf{x}, \mathbf{u}, \xi, t)$$

conservative variables and flux can be obtained by

$$\begin{pmatrix} \rho \\ \rho \mathbf{U} \\ \rho E \end{pmatrix} = \int \psi f d\Xi, \quad \mathbf{F}(t) = \int u \psi f d\Xi$$

gas distribution function modeled by K. Xu

$$f = e^{-t/\tau} g^l [1 - (\tau + t) a_x^l \cdot u - \tau A^l] \mathbb{H}(u)$$
$$+ e^{-t/\tau} g^r [1 - (\tau + t) a_x^r \cdot u - \tau A^r](1 - \mathbb{H}(u))$$
$$+ (1 - e^{-t/\tau}) g^c + [(t + \tau) e^{-t/\tau} - \tau] a_x^c \cdot u g^c + (t - \tau + \tau e^{-t/\tau}) A^c g^c$$

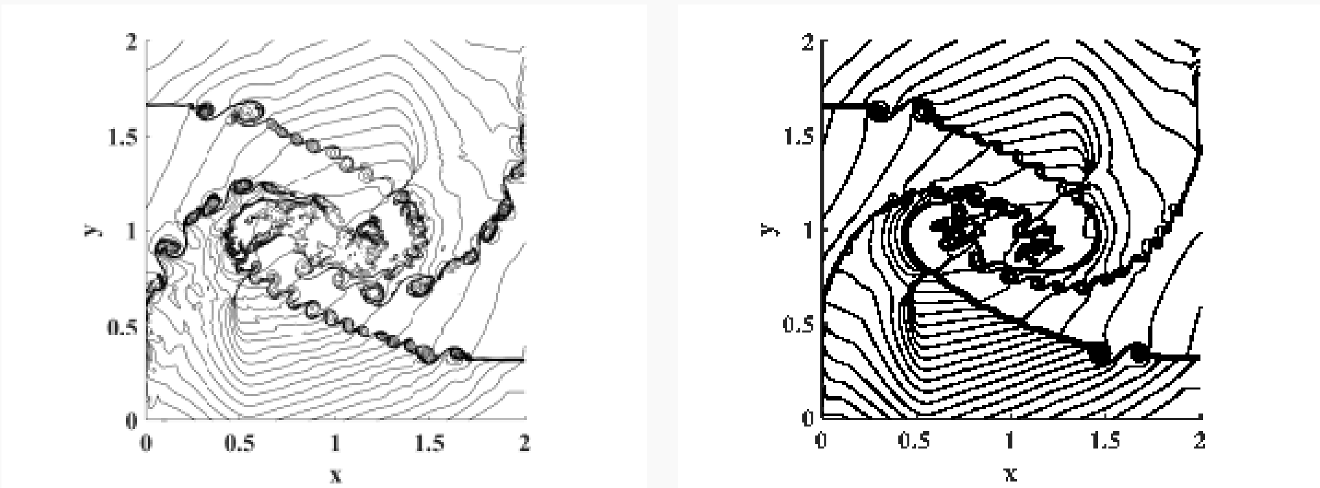
flux function gives

$$\mathrm{F}(t) = \mathrm{F}(\mathrm{W}) + \mathrm{F}_t((\mathrm{W}, \nabla \mathrm{W}))t$$

two-stage forth order (S2O4) temporal advance($t^n \longrightarrow t^{n+1}$)

$$\mathrm{W}^* = \mathrm{W}^n + \frac{1}{2} \Delta t \mathcal{L}(\mathrm{W}^n) + \frac{1}{8} \Delta t^2 \frac{\partial}{\partial t} \mathcal{L}(\mathrm{W}^n)$$
$$\mathrm{W}^{n+1} = \mathrm{W}^n + \Delta t \mathcal{L}(\mathrm{W}^n) + \frac{1}{6} \Delta t^2 \left(\frac{\partial}{\partial t} \mathcal{L}(\mathrm{W}^n) + 2 \frac{\partial}{\partial t} \mathcal{L}(\mathrm{W}^*) \right)$$

A simple test: Configuration 6



(a) S2O4+GKS

(b) SSP-RK+L-F

Figure 1: Configuration 6: the density distributions at $t = 1.6$ with 800×800 meshes. (a-b) different schemes.

Resolution $\Leftrightarrow$ Robustness

WENO-AO RECONSTRUCTION

For the fifth-order WENO-AO method, the three sub-stencils give

$$\mathbb{S}_1^{r3} = \{I_{i-2}, I_{i-1}, I_i\}, \quad \mathbb{S}_2^{r3} = \{I_{i-1}, I_i, I_{i+1}\}, \quad \mathbb{S}_3^{r3} = \{I_i, I_{i+1}, I_{i+2}\}$$

The large stencil gives

$$\mathbb{S}^{r5} = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$$

$p_1^{r3}(x), p_2^{r3}(x), p_3^{r3}(x)$ and $p^{r5}(x)$ are reconstruct polynomials correspond to individual stencil, and

$$\frac{1}{h}\int_{I_{i+j}} p^{r5}(x)dx = \overline{W}_{i+j}, \quad j = 0, 1, 2,$$
$$\frac{1}{h}\int_{I_{i+j+k-3}} p_k^{r3}(x)dx = \overline{W}_{i+j+k-3}, \quad j = 0, 1, 2, \ k = 1, 2, 3.$$

WENO-AO RECONSTRUCTION

By the given linear weights

$$d_3 = d_{H_i}; \quad d_0 = (1 - d_{Hi})(1 - d_{Lo})/2; \quad d_1 = (1 - d_{Hi})d_{Lo}; \quad d_2 = d_0$$

and calculate the smoothness indicators(computationally expensive)

$$\beta_k = \sum_{q=1}^{q_k} h^{2q-1} \int_{x_{i-1/2,i}}^{x_{i+1/2,i}} \left(\frac{d^q}{dx^q} p_k(x) \right)^2 dx$$

the non-linear weights (WENO-Z type)and normalized weights give

$$\omega_k = d_k \left(1 + \left(\frac{\tau_Z}{\beta_k + \varepsilon} \right)^2 \right) \Rightarrow \overline{\omega}_k = \sum_{d=0}^{3} \omega_d$$

the final form of the reconstruct polynomial gives

$$P(x) = \overline{\omega}_3 \left(\frac{1}{d_3} p_3^{r5}(x) - \sum_{k=0}^{2} \frac{d_k}{d_3} p_k^{r3}(x) \right) + \sum_{k=0}^{2} \overline{\omega}_k p_k^{r3}(x)$$

Robustness Analysis

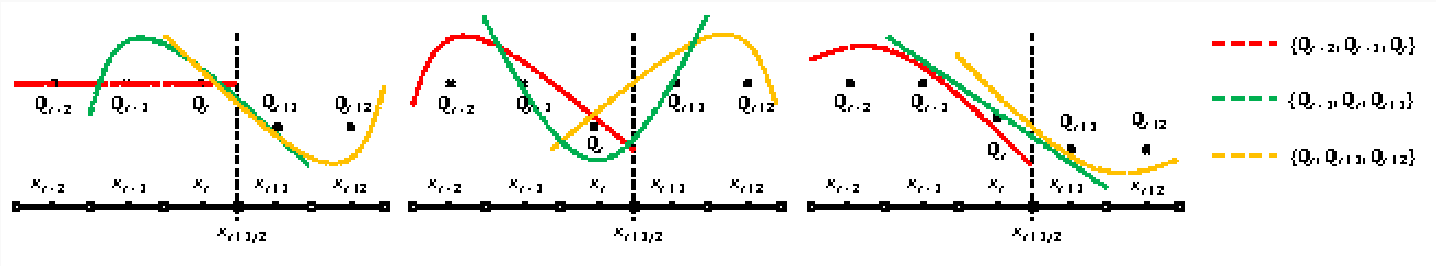
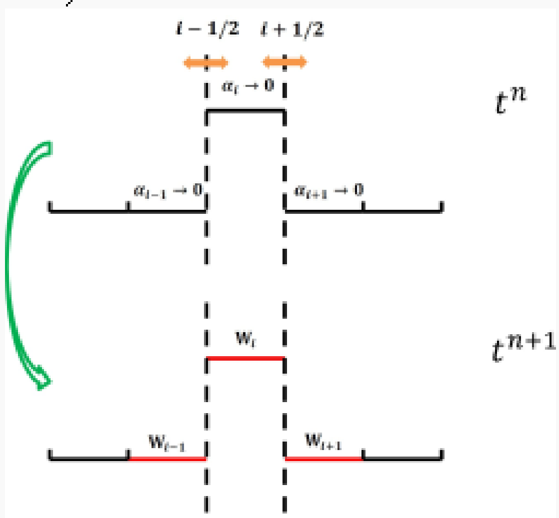


Figure 2: Three possible distributions of variables $Q_{i-2}, \cdots, Q_{i+2}$. (a) Smooth sub-stencils exist, and the WENO reconstruction will automatically approximate to $\{Q_{i-2}, Q_{i-1}, Q_i\}$ by weights. (b-c) Each sub-stencils have a discontinuity, while WENO can only select the relatively smooth sub-stencil by weights, which means the effectiveness of the reconstruction polynomial for each sub-stencil is reduced.

DISCONTINUITY FEEDBACK FACTOR

The idea of DF is to start from the interface reconstruction values at the current time step, and predict the cell that the discontinuities will enter at the next step. Denote $\alpha_{ij} \in (0, 1]$ as DF at a cell Ω_{ij}

- $\alpha_{ij} = \prod_{p=1}^{4} \prod_{m=1}^{2} \alpha_{p,m}$
- $\alpha_{p,m} = \frac{1}{1+D^2}$ is the DF corresponding to the Gaussian point $\mathbf{x}_{p,m}$
- $D = \frac{|p^l - p^r|}{p^l} + \frac{|p^l - p^r|}{p^r} + \left(\mathrm{Ma}_n^l - \mathrm{Ma}_n^r\right)^2 + \left(\mathrm{Ma}_t^l - \mathrm{Ma}_t^r\right)^2$



Strong discontinuities lead to $\alpha_{i,j}$ approximate to 0.

HYBRID RECONSTRUCTION

KEY: Combine with the WENO-AO & DF factor

step 1: calculate the $\alpha_{i,j}$ for cell $\Omega_{i,j}$ by $\alpha_{ij} = \prod_{p=1}^{4} \prod_{m=1}^{2} \alpha_{p,m}$

step 2: shock capture by $\min(\alpha_{i,j}, \alpha_{threshold})$

$$\text{if } \min(\alpha_{i,j}, \alpha_{threshold}) = \alpha_{threshold} \Rightarrow \text{go to step3}$$
$$\text{if } \min(\alpha_{i,j}, \alpha_{threshold}) = \alpha_{i,j} \Rightarrow \text{go to step4}$$

step 3: spatial reconstruction by WENO-AO

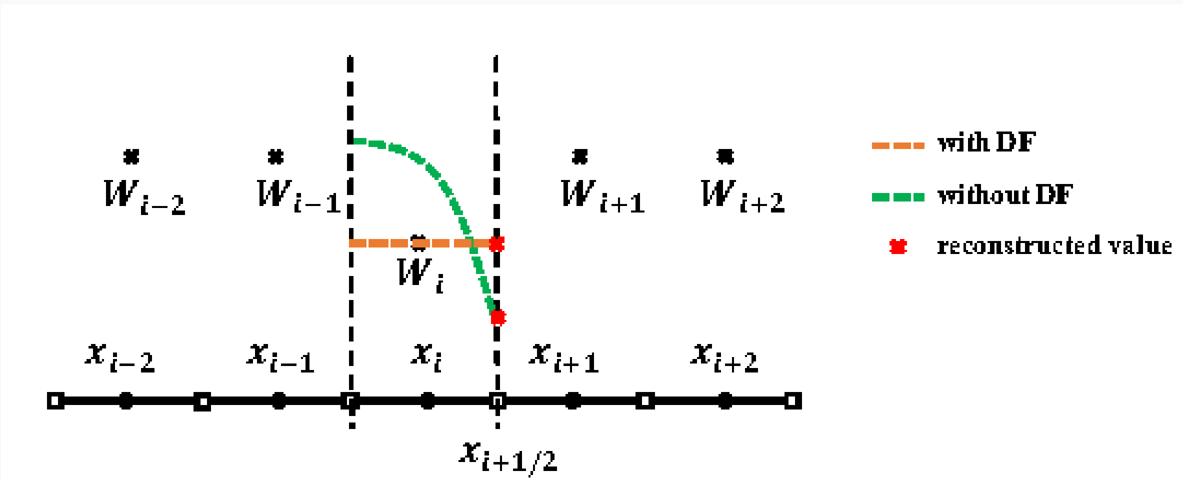
step 4: spatial reconstruction by Hybrid: only stencil $\mathbb{S}_2^{r3} = \{I_{-1}, I_0, I_1\}$ **is used, extra modification is needed for reconstruction polynomial**

Initial polynomial (for stencil $\mathbb{S}_2^{r3}$) gives (zero-mean form)

$$p_2^{r3}(x) = W_{i,j} + \frac{1}{\Delta x}(W_{i+1,j} - W_{i,j})(x - c_0) + \frac{1}{\Delta x^2}\left(\frac{1}{2}W_{i-1,j}W_{i,j} + \frac{1}{2}W_{i+1,j}\right)(x^2 - c_1)$$

modified by DF factor

$$p_{DF}^{r3}(x) = W_{i,j} + \alpha_{i,j}\left(p_2^{r3}(x) - W_{i,j}\right)$$

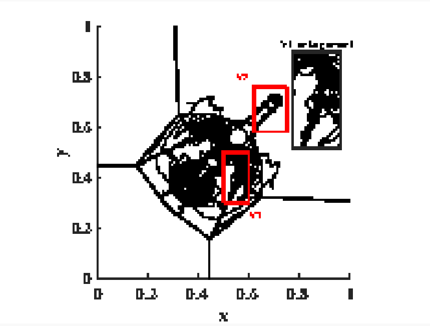


$\alpha_{threshold}$ closely related to discontinuity capture

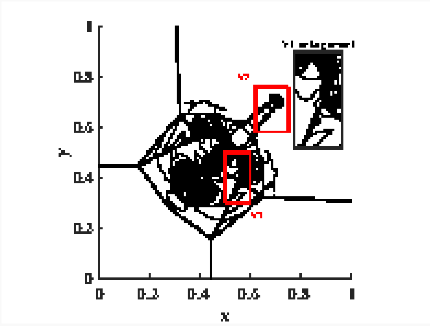
- $\alpha_{threshold}$ $\uparrow$, more "HYBRID stencils" are used
  robustness$\uparrow$, resolution$\downarrow$
- $\alpha_{threshold}$ $\downarrow$, more "WENO-AO stencils" are used
  robustness$\downarrow$, resolution$\uparrow$

KEY: how to select a suitable value of $\alpha_{threshold}$

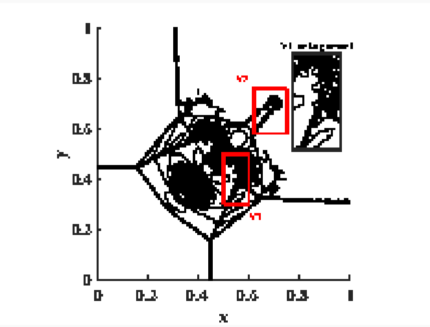
Relationship between $\alpha_{threshold}$ & resolution



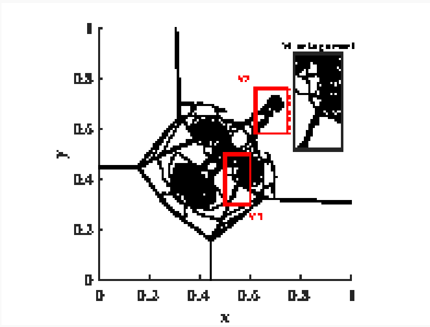
(a) $\alpha_{thres} = 0.1$



(b) $\alpha_{thres} = 0.3$



(c) $\alpha_{thres} = 0.5$



(d) $\alpha_{thres} = 0.8$

(a-d) Different values of α_{thres} using S2O4 GKS solver

Relationship between $\alpha_{threshold}$ & robustness

Hurricane-like problem: Rarefaction wave.
Maximum Mach number the algorithm can calculate: $Ma = \frac{|u_0|}{c}$

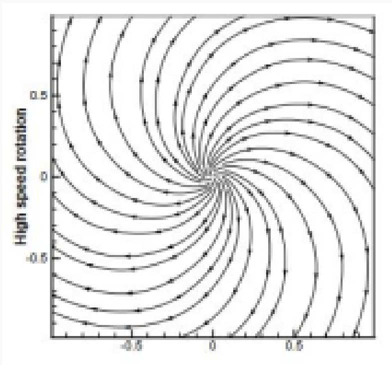
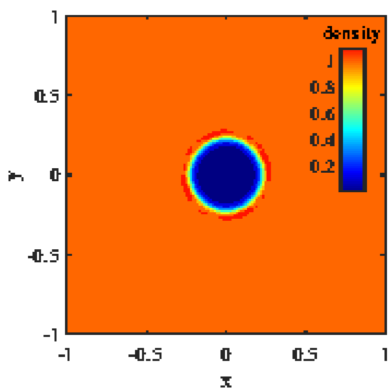


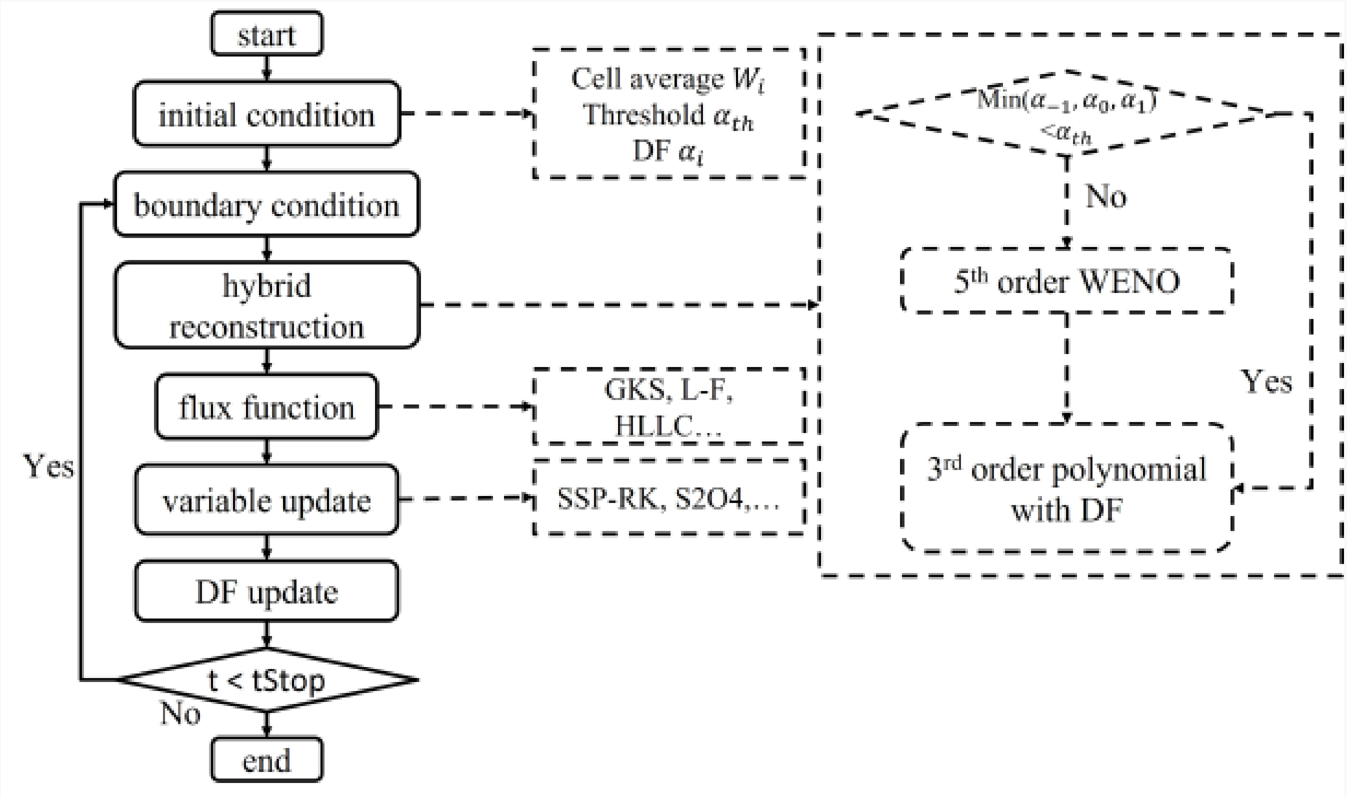
Table 1: Maximum Mach number using different values of α_{thres}.

GKS solver	Maximum Mach number
$\alpha_{thres} = 0.1$	8.4
$\alpha_{thres} = 0.3$	13.2
$\alpha_{thres} = 0.5$	14.0
$\alpha_{thres} = 0.8$	14.5

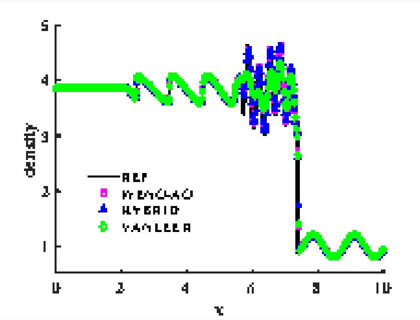


$\alpha_{threshold} = 0.5$ **is used.**

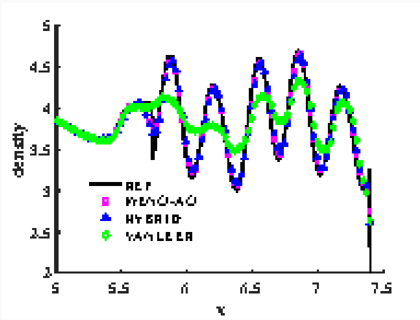
Algorithm of robustness-enhanced high-order FV scheme



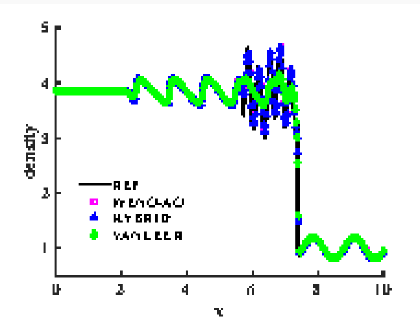
Shu-Osher problem



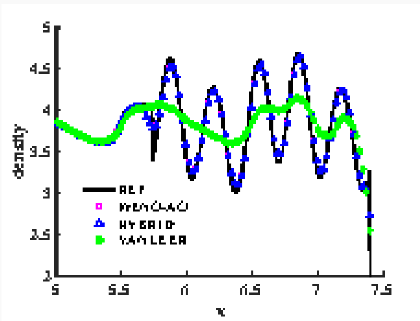
(g) GKS, Global



(h) GKS, Local



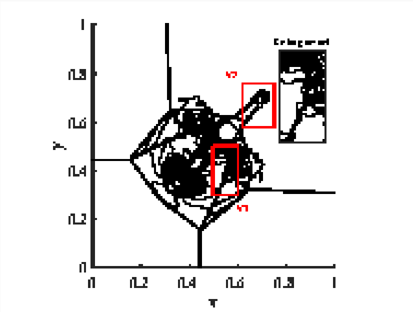
(i) L-F, Global



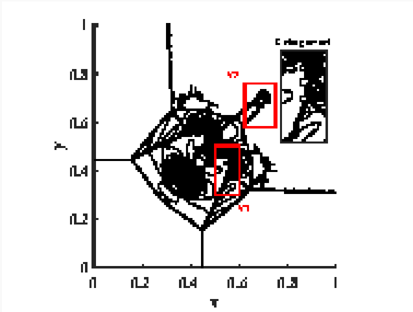
(j) L-F, Local

- density distribution at $t = 1.8$
- cell size $\Delta x = 1/40$

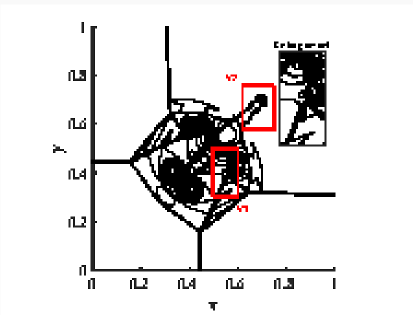
Configuration 3



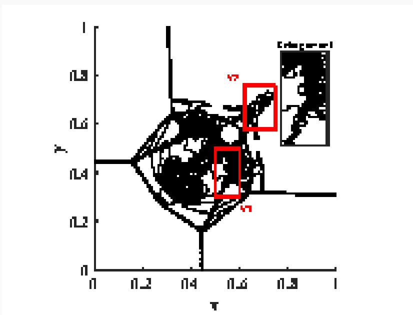
(k) GKS, WENO-AO



(l) GKS, Hybrid



(m) L-F, WENO-AO



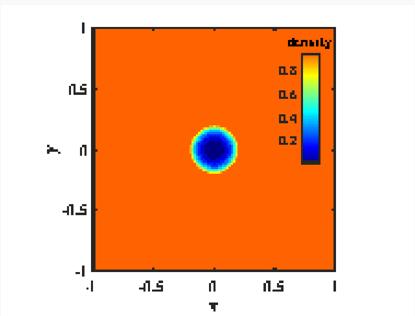
(n) L-F, Hybrid

- density distribution at $t = 0.6$
- cell size $\Delta x = 1/500$

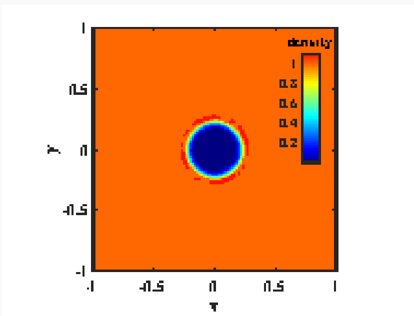
Hybrid reconstruction keeps high resolution.

NUMERICAL RESULTS

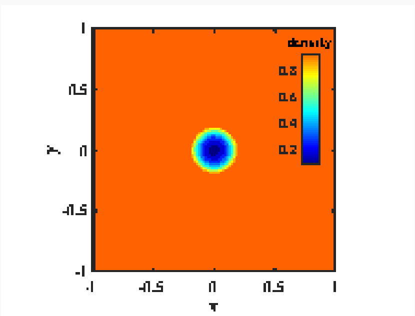
Hurricane-like problem



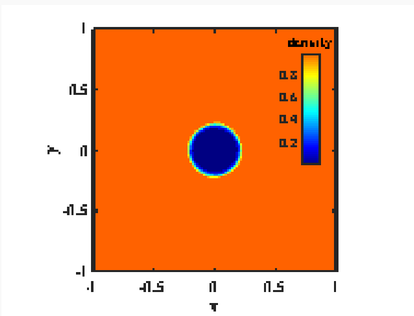
(o) GKS, WENO-AO, Ma 2.0



(p) GKS, Hybrid, Ma 16.0



(q) L-F, WENO-AO, Ma 1.4



(r) L-F, Hybrid, Ma 7.0

Table 2: Hurricane-like problem: Maximum Mach number using different reconstruction methods

GKS solver	Maximum Mach
WENO-AO	2.0
Hybrid	16.0
LF solver	**Maximum Mach**
WENO-AO	1.4
Hybrid	7.0

DF can significantly improve algorithm's robustness.

DF distribution

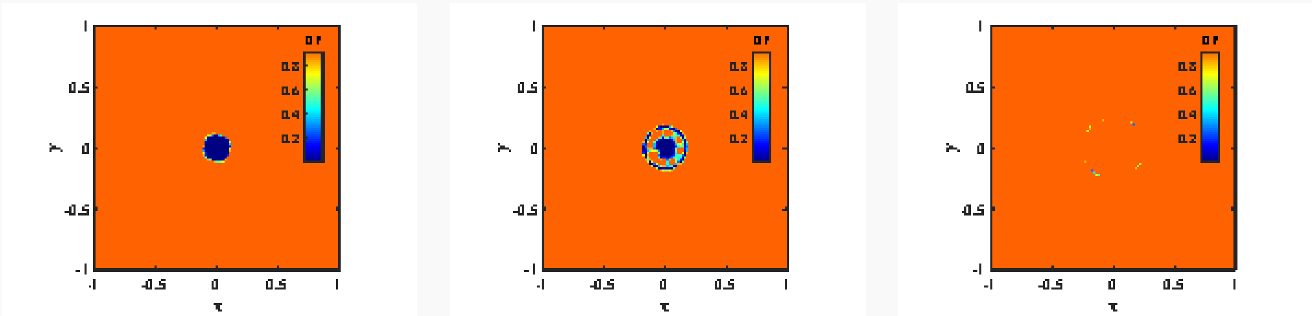


Figure 3: DF distribution at time step 15, 30, 45 (from left to right). GKS solver is used.

DF performs well in capturing discontinuity.

High-mach number astrophysical jet

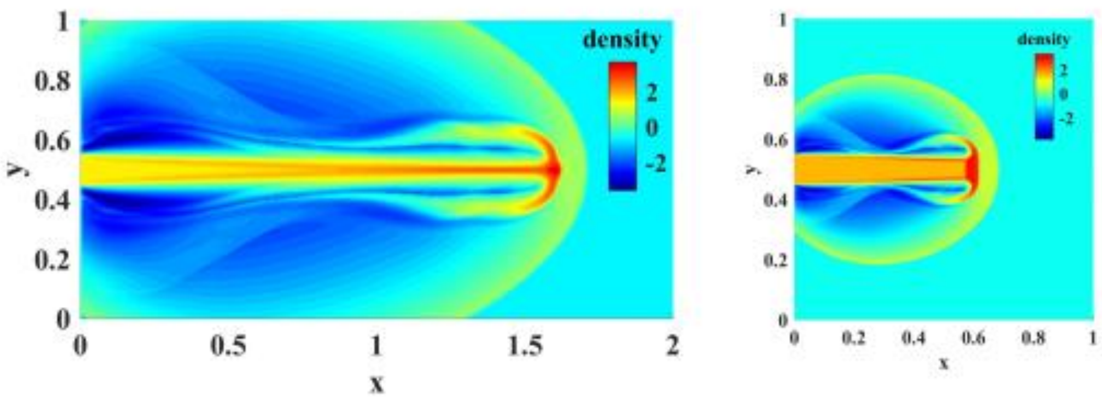


Figure 4: Density distribution (non-linear function of $\phi = \log(\rho)$). Left: Mach 80, Right: Mach 20000. L-F solver is used.

Approaching positivity-preserving scheme.

SUMMARY & FUTURE WORK

Summary

- The DF-based reconstruction is robust for hypersonic flow
- The DF-based reconstruction keeps high resolution compared to classical WENO method
- The DF-based reconstruction works with multiple schemes(like GKS, Lax-Friedrichs, etc.)

Future work

- More efficient and robust DF-based scheme will be conducted based on compact GKS/DG on unstructured meshes.

Thanks for your attention

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Collaborate with:
Prof. Xing Ji Team in XJTU
Prof. Kun Xu Team in HKUST