Oral presentation | Incompressible/compressible/hypersonic flow
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Fri. Jul 19, 2024 10:45 AM - 12:45 PM Room D

[13-D-03] Determining the spatial resolution in direct numerical simulations of compressible turbulence by using Burgers equation

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Determining the spatial resolution in direct numerical simulations of compressible turbulence by using Burgers equation

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Outline

1. Introduction
2. Characteristic scales of shock wave in Burgers turbulence
3. Extension to compressible Navier-Stokes turbulence
4. Conclusion

1. Introduction

1. Introduction

The spatial resolution of DNS need to be fine enough to cover the minimum scale of fluid structure.

➢For the **incompressible turbulence**, the average Kolmogorov scale $\eta_{avg} = \left(\langle \nu \rangle^3 / \left\langle \nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle \right)^{1/4}$ is considered as the minimum scale[1].

➢For the **compressible turbulence**, studies[2] usually consider:

- Minimum scale → the average Kolmogorov scale
- Spatial resolution criterion:

$$\eta k_{max} > 1.5 \xLeftrightarrow{k_{max}=\frac{\pi}{\Delta x}} \frac{\eta}{\Delta x} > 0.5$$

k_{max} : the largest wavenumber Δx: the grid length

[1] Pope 2000 *Turbulent Flows*
[2] Samtaney et al. PoF 2001, Petersen & Livescu PoF 2010, Wang et al. 2012 JFM, Wang et al. PRF 2017

1. Introduction

Sometimes, this average Kolmogorov scale is not fine enough:

➢Incompressible turbulence:

Cases with strong intermittency require a more stringent requirement:

$$k_{max}\eta_{min} > 1$$

with η_{min} the minimum local Kolmogorov scale[1].

[1] Meneveau & Sreenivasan JFM 1991, Sreenivasan FTaC 2004, Yakhot & Sreenivasan J. Stat. Phys. 2005, Watanabe & Gotoh JFM 2007

1. Introduction

Sometimes, this average Kolmogorov scale is not fine enough:

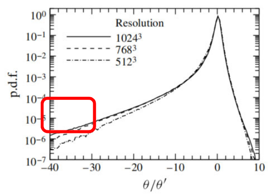
➢Compressible turbulence:

A non-convergence of velocity gradient skewness and p.d.f of dilatation is observed in current numerical studies using this classical criterion.

Example[1]:

Resolution	Re_λ	M_t	$k_{max}\eta$	u'	L_f	L_f/η	$\langle\epsilon\rangle$	θ'	ω'	S_3
512^3	256	1.02	1.65	2.15	1.45	224	0.529	7.3	22.9	−1.4
768^3	258	1.00	2.47	2.13	1.46	226	0.533	7.7	22.5	−1.9
1024^3	254	1.03	3.33	2.16	1.46	225	0.535	7.9	22.8	−2.2

Simulation parameters and resulting flow statistics. [1]



The p.d.f.s of the normalized dilatation at three grid resolutions[1]

Analysis on these anomalies[1]:

- *Due to the regions of* ***strong shocklets*** *that are not directly resolved.*
- ***May not influence the global quantity,*** *as the conservation laws across the shocks are still exactly satisfied.*

[1] Wang et al. JFM 2012

1. Introduction

A thorough understanding of energy transfer in compressible turbulence may require a study on **skewness, 2nd and 3rd order moments of velocity gradient** etc. [1]

- 2nd order: energy related quantities(such as enstrophy $\langle\omega^2\rangle$)
- 3rd order: transfer related quantities(such as vortex stretching $\langle\omega_i S_{ij}\omega_j\rangle$)

Consequently, these values need to be accurately calculated, while the classical spatial resolution criterion can not give convergent result.

Thus, for compressible turbulence

- What is the minimum scale?
- What is the spatial resolution requirement for the DNS?

[1] Tsinober 2009 *An Informal Conceptual Introduction to Turbulence*, Fang et al. PLA 2015, Liu, Fang & Shao Chin. Physics B 2020, Buaria, Pumir & Bodenschatz PRF 2020, Yang et al. JFM 2022

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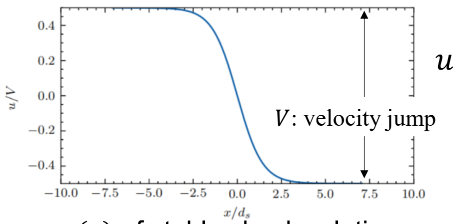
2. Characteristic scales of shock wave in Burgers turbulence

2.1 Burgers equation

- Why study shock? Because the anomalies come from the shocklet region.
- Good tool to study shock wave:

$$\text{Burgers equation[1]: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$$

Stable shock solution



$u(x)$ of stable shock solution

$$u = -\frac{1}{2}V\tanh\left[\frac{V}{4\nu}x\right] \longrightarrow u = -\frac{1}{2}V\tanh\left[\frac{x}{\sqrt{2}d_s}\right]$$

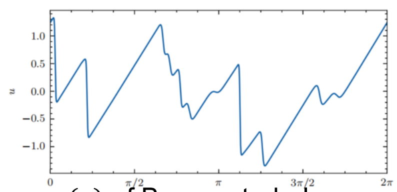
Characteristic scale[2]:
$d_s = 2\sqrt{2}\nu/V$

[1] Burgers Trans. Roy. Neth. Acad. Sci. Amsterdam 1939 [2] Benton & Platzman Q. Appl. Math 1972

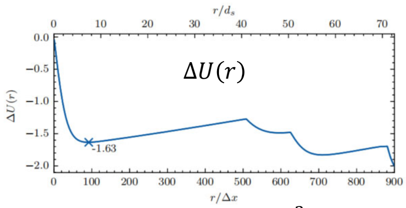
2.2 Burgers turbulence

Stable shock solution for Burgers equation: characteristic scale $d_s = 2\sqrt{2}\nu/V$

- In Burgers turbulence, how to get this characteristic scale?



$u(x)$ of Burgers turbulence



$d_s = 8.66 \times 10^{-3}$

Measure the maximum velocity jump across shock!

- In Burgers turbulence, shocks all give negative velocity increment.

$$\Delta U(r) = \min_{x\in\mathbb{R}}[u(x+r) - u(x)], r > 0$$ → Measure the negative velocity jump

First local minimum of $\Delta U_{\min,1st}$ → Measure the maximum velocity jump across one shock!

Characteristic scale: $d_s = -2\sqrt{2}\nu/\Delta U_{\min,1st}$

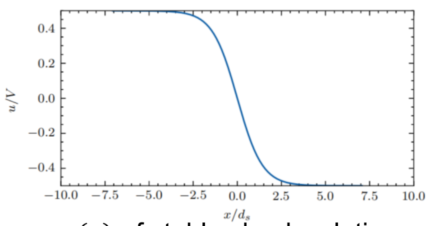
Weakness:
- Unable to locate where the minimum scale is.
- Unable to show the scales of other shocks!

2.1 Burgers equation

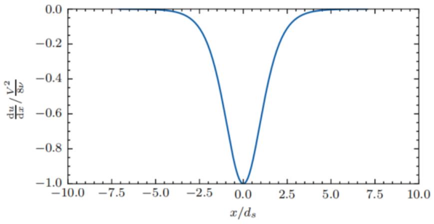
- Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$

Stable shock solution

$$u(x) = -\frac{1}{2}V\tanh\left[\frac{x}{\sqrt{2}d_s}\right], d_s = \frac{2\sqrt{2}\nu}{V} \Longrightarrow \theta(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\nu}{d_s^2}\left[1 - \tanh^2\left(\frac{x}{\sqrt{2}d_s}\right)\right]$$



$u(x)$ of stable shock solution



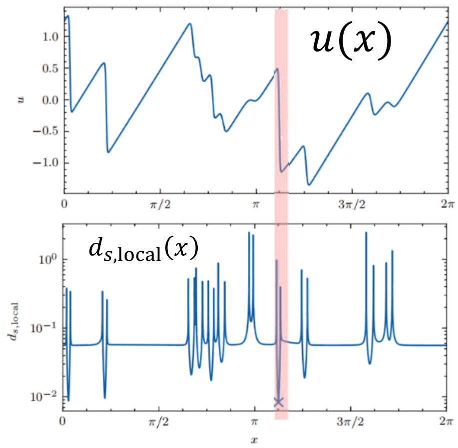
$\theta(x)$ of stable shock solution

Local characteristic scale:

$$\boldsymbol{d_{s,\mathrm{local}}(x) = \sqrt{\frac{\nu}{|\theta(x)|}}}$$

The local minimum is a shock.

2.2 Burgers turbulence



Local characteristic scale: $d_{s,\mathrm{local}}(x) = \sqrt{\nu/|\theta(x)|}$

Characteristic shock scale: $d_s = \min_x d_{s,\mathrm{local}}(x)$

Method	For the example in left figure (numerical results)
By velocity increment $d_s = -2\sqrt{2}\nu/\Delta U_{\min,1st}$	8.66×10^{-3}
By local scale $d_s = \min_x d_{s,\mathrm{local}}(x)$	8.27×10^{-3}

- Of same magnitude
- Smaller for the one obtained by local scale

The minimum scale of Burgers turbulence is $d_s = \min_x \sqrt{\nu/|\theta(x)|}$.

2.3 Spatial resolution requirement

- Discretization influence:
  - The calculation of minimum scale → 2.3.1
  - The calculation of skewness, 2nd and 3rd moments of velocity gradient. → 2.3.2
- Besides, the level of influence depends on the discretization method.
  - The 2nd order central difference.
  - The 6th order central difference.

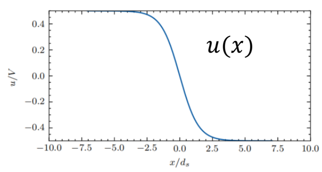
Theoretical solution $u(x)$ → Discretized solution $u_n(\phi) = u[(n+\phi)\Delta x]$ → (General expression of) central difference $\theta_{n,\text{num}}(\phi) = \frac{\sum_{m=-N}^{N} a_m u_{n+m}(\phi)}{\Delta x}$

ϕ: calculation phase, random, $\phi \in [-\frac{1}{2}, \frac{1}{2})$

Example: 2nd order central difference: $\theta_{n,2\text{nd}}(\phi) = \frac{1}{2} u_{n+1}(\phi) - \frac{1}{2} u_{n-1}(\phi)$

2.3.1 Influence on minimum scale

- Stable shock solution: $u(x) = -\frac{1}{2} V \tanh\left[\frac{x}{\sqrt{2} d_s}\right]$.



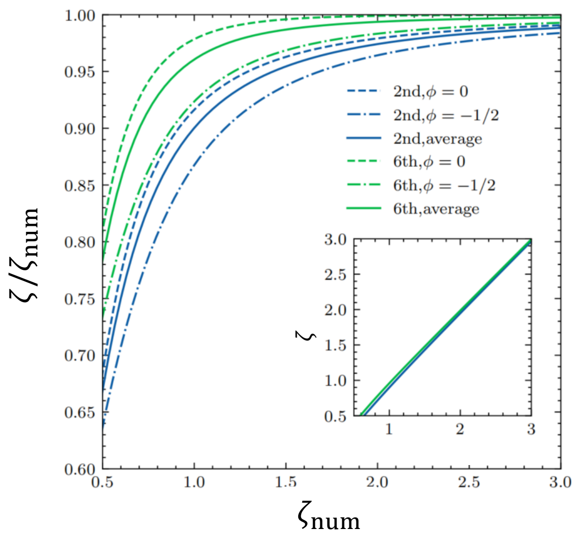
- Theoretical: d_s
- Numerical:
  - The numerical maximum velocity gradient: $|\theta|_{max,\text{num}}(\phi) = \frac{-\sum_{m=-N}^{N} a_m u_m(\phi)}{\Delta x}, (i.e. n = 0)$
  - The numerical shock scale: $d_{s,\text{num}}(\phi) = \sqrt{\frac{d_s \Delta x}{\sqrt{2} \sum_{m=-N}^{N} a_m \tanh(\phi \Delta x / \sqrt{2} d_s)}}$
  - The numerical resolution ratio: $\zeta_{\text{num}}(\phi) = \sqrt{\frac{\zeta}{\sqrt{2} \sum_{m=-N}^{N} a_m \tanh(\phi / \sqrt{2} \zeta)}}$

What matters is the ratio! $\zeta = d_s / \Delta x$

- The relation between numerical and true(theoretical) minimum scale:

$$\frac{d_s}{d_{s,\text{num}}}(\phi) = \frac{\zeta}{\zeta_{\text{num}}}(\phi) = \sqrt{\sqrt{2}\zeta \sum_{m=-N}^{N} a_m \tanh\left(\phi / \sqrt{2}\zeta\right)}$$

2.3.1 Influence on minimum scale



- Stable shock solution: $u(x) = -\frac{1}{2}V\tanh\left[\frac{x}{\sqrt{2}d_s}\right]$.

The relation between numerical and true(theoretical) minimum scale:

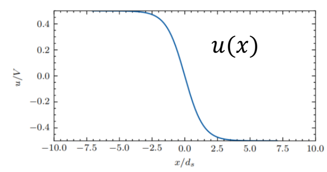
$$\frac{d_s}{d_{s,\text{num}}}(\phi) = \frac{\zeta}{\zeta_{\text{num}}}(\phi) = \sqrt{\sqrt{2}\zeta\sum_{m=-N}^{N} a_m \tanh\left(\phi/\sqrt{2}\zeta\right)}$$

Example: 2[nd] order central difference:

$$\frac{d_s}{d_{s,\text{num}}}(\phi) = \frac{\zeta}{\zeta_{\text{num}}}(\phi) = \sqrt{\frac{\sqrt{2}\zeta}{2}\tanh\left(\frac{\phi+1}{\sqrt{2}\zeta}\right) - \frac{\sqrt{2}\zeta}{2}\tanh\left(\frac{\phi-1}{\sqrt{2}\zeta}\right)}$$

$\zeta_{\text{num}} - \zeta/\zeta_{\text{num}}$ plot (Inset: $\zeta_{\text{num}} - \zeta$ plot)
for 2[nd] and 6[th] order central difference
at different phases and average for phase($\mathbb{E}_{\phi\sim\mathfrak{U}[-0.5,0.5)}\left[\frac{\zeta}{\zeta_{\text{num}}}(\phi)\right]$)

2.3.2 Influence on vel. grad. moment



- Stable shock solution: $u(x) = -\frac{1}{2}V\tanh\left[\frac{x}{\sqrt{2}d_s}\right]$.

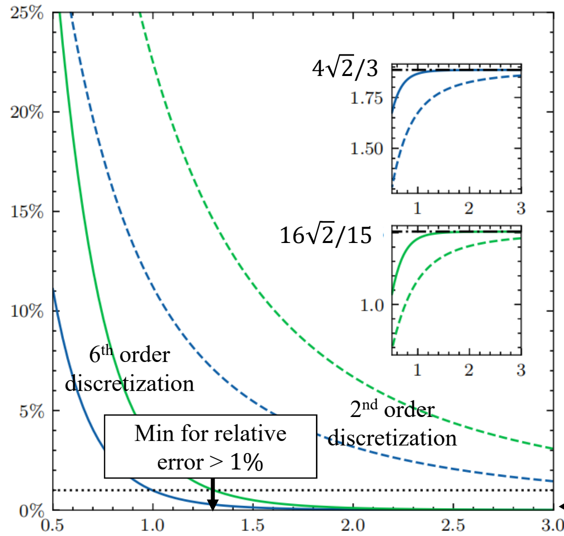
A infinite solution→ we take the integral as the "moment"

- **Theoretical:**
  - 2[nd] order moment: $\int_{-\infty}^{+\infty}\theta^2(x)\mathrm{d}x = \frac{4\sqrt{2}\nu^2}{3d_s^3}$
  - 3[rd] order moment: $\int_{-\infty}^{+\infty}\theta^3(x)\mathrm{d}x = -\frac{16\sqrt{2}\nu^3}{15d_s^5}$
- **Numerical:**
  - 2[nd] order moment:

$$\sum_{n=-\infty}^{+\infty}\Delta x\theta_{n,\text{num}}^2(\phi) = 2\zeta\sum_{n=-\infty}^{+\infty}\left[\sum_{m=-N}^{N} a_m\tanh\left(\frac{n+m+\phi}{\sqrt{2}\zeta}\right)\right]^2\frac{\nu^2}{d_s^3} = f_2(\zeta,\phi)\frac{\nu^2}{d_s^3}$$

  - 3[rd] order moment:
    - $\sum_{n=-\infty}^{+\infty}\Delta x\theta_{n,\text{num}}^3(\phi) = -2\sqrt{2}\zeta\sum_{n=-\infty}^{+\infty}\left[\sum_{m=-N}^{N} a_m\tanh\left(\frac{n+m+\phi}{\sqrt{2}\zeta}\right)\right]^3\frac{\nu^3}{d_s^5} = -f_3(\zeta,\phi)\frac{\nu^2}{d_s^3}$

2.3.2 Influence on vel. grad. moment

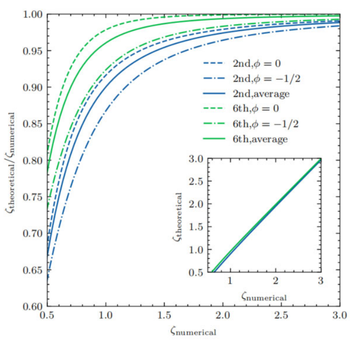


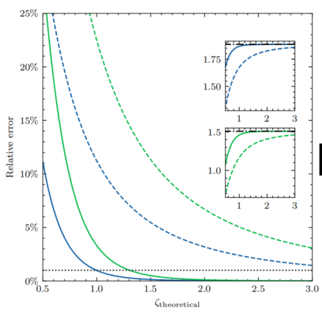
	Theoretical	Numerical
2nd order moment	$\frac{4\sqrt{2}\nu^2}{3d_s^3}$	$f_2(\zeta,\phi)\frac{\nu^2}{d_s^3}$
3rd order moment	$-\frac{16\sqrt{2}\nu^3}{15d_s^5}$	$-f_3(\zeta,\phi)\frac{\nu^2}{d_s^3}$

ζ – Relative error plot, Up Inset : $\zeta - f_2(\zeta)$ plot, Below inset : $\zeta - f_3(\zeta)$ plot
Green: 3rd order moment, Blue: 2nd order moment
Solid line: 6th order cen. diff.
Dashed line: 2nd order cen. diff.
Average for phase($\phi \sim \mathfrak{U}[-0.5,0.5)$)

2.3 Numerical criterion

- Discretization influence:
  - The calculation of minimum scale → 2.3.1
  - The calculation of skewness, 2nd and 3rd moments of velocity gradient. → 2.3.2





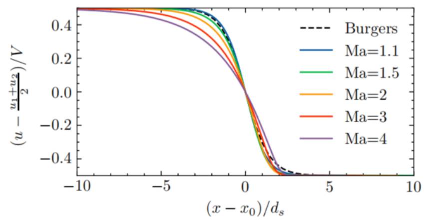
For 6th order central difference, if we want the 2nd and 3rd order moment has relative error < 1%:

$$\zeta_{\text{num}} = \frac{d_{s,\text{num}}}{\Delta x} \gtrsim 1.5$$

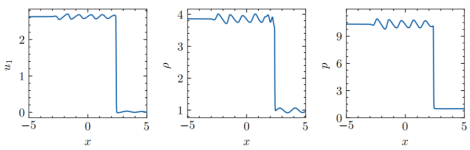
3. Extension to Navier-Stokes turbulence

3.1 1D compressible turbulence

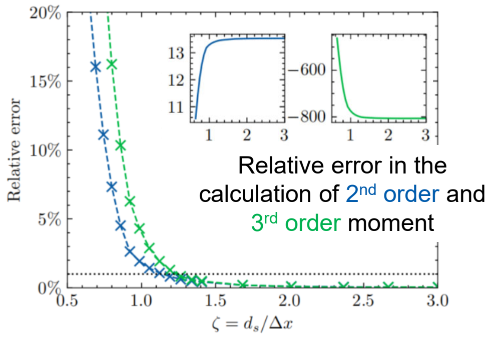
1. If the inlet Mach number is not so big, the velocity profile of shock resembles a lot to Burgers shocks.



2. Validation on the 1D compressible Shu-Osher shock tube problem



Solution of Shu-Osher shock tube problem



Relative error in the calculation of 2nd order and 3rd order moment

3.2 Extension to multidimensional

- Local characteristic scale: $d_{s,local}(x) = \sqrt{\nu/|\theta(x)|}$
- Average Kolmogorov scale: $\eta_{avg} = \left(\langle\nu\rangle^3 \Big/ \left\langle \nu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle\right)^{1/4}$
- Local Kolmogorov scale: $\eta(\boldsymbol{x}) = \left(\nu(\boldsymbol{x})^2 / \frac{\partial u_i}{\partial x_j}(\boldsymbol{x})\frac{\partial u_i}{\partial x_j}(\boldsymbol{x})\right)^{1/4}$
  - In incompressible, $\eta(x) = \sqrt{\nu(x)/|\boldsymbol{\omega}(x)|}$
  - In 1D compressible, $\eta(x) = \sqrt{\nu(x)/|\theta(x)|}$
  - In 2D&3D compressible, $\eta(x) = \sqrt{\nu(x)/\sqrt{\boldsymbol{\omega}^2(x) + \theta^2(x)}}$

- **Conclusion: Local Kolmogorov scale is a scale including both vortex magnitude and shock magnitude.**

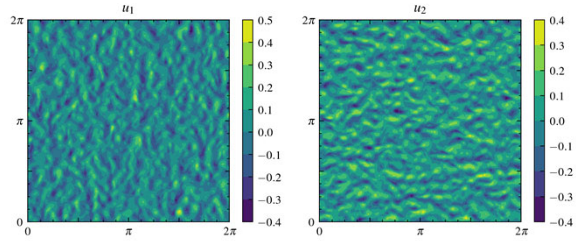
3.2 Extension to multidimensional

What is the relation between local Kolmogorov scale and the local shock scale?

We do the simulation of a 2D compressible flow.

- Resolution: 8192^2
- Other detail configuration: presented later, cf. slide P26 (case 1)

The flow field of a fully developed compressible turbulence



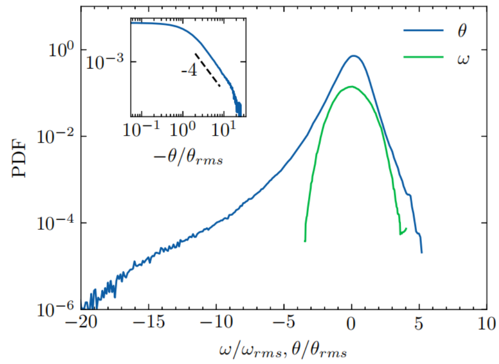
$\mathrm{Ma}_t = 0.35$

Calculate:

- The p.d.f. of θ and $\omega = |\boldsymbol{\omega}|$
- The p.d.f. of local Kolmogorov scale η, vortex characteristic scale $\sqrt{\nu(x)/|\boldsymbol{\omega}(x)|}$ and shock characteristic scale $\sqrt{\nu(x)/|\theta(x)|}$

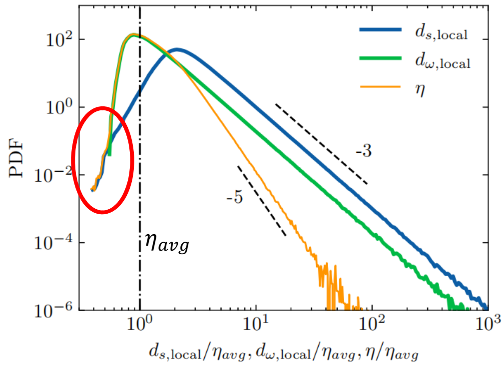
3.2 Extension to multidimensional

What is the relation between local Kolmogorov scale and the local shock scale?



p.d.f. of θ and $\omega = |\boldsymbol{\omega}|$

θ has much larger extreme value than ω. Consequently, the shock has smaller structure than the vortex.



p.d.f. of η, $\sqrt{\nu(x)/|\boldsymbol{\omega}(x)|}$ and $\sqrt{\nu(x)/|\theta(x)|}$

The minimum of local shock scale is smaller than the local vortex scale. In small scales, the local Kolmogorov scale coincide with the local shock scale.

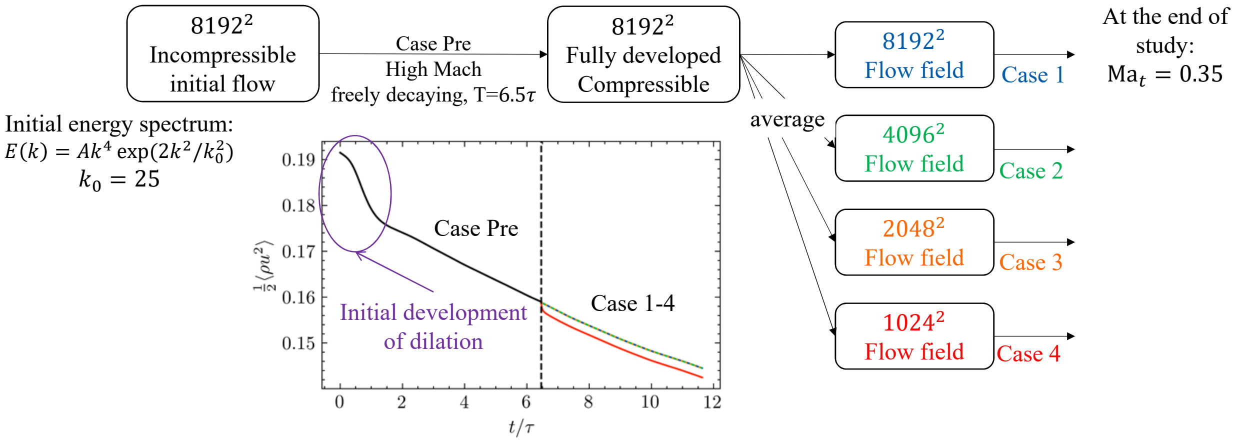
3.2 Extension to multidimensional

What is the relation between local Kolmogorov scale and the local shock scale?

- **Conclusion: The minimum scale of turbulence should be the minimum scale of local Kolmogorov scale. In compressible turbulence, it is determined by the strongest shock.**
- In 1D shock tube study, the criterion is $d_s/\Delta x \geq 1.5$.
- **Consequently, in compressible turbulence, the criterion of numerical resolution is $\eta_{min}/\Delta x \geq 1.5$**

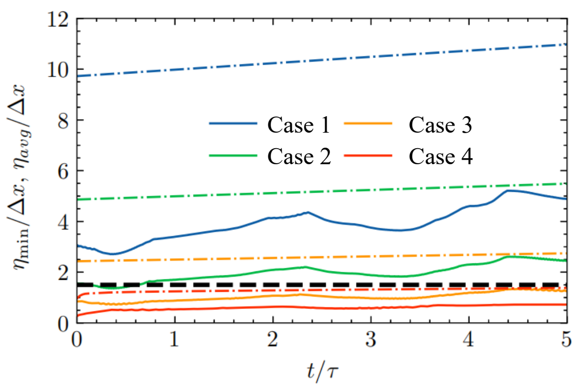
3.3 Validation of numerical criterion

- Solver: an open-source solver ASTR[1], which uses 6th order compact method for convection term, no shock capturing used, no forcing(freely decaying)



[1]https://github.com/astr-code/astr

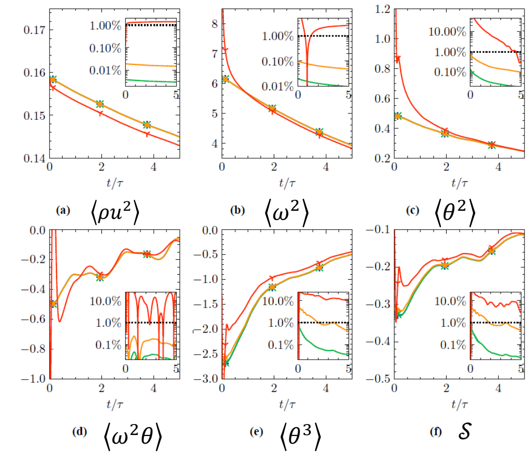
3.3 Validation of numerical criterion



Evolution of η_{avg} and η_{min} for cases 1-4
Solid line: η_{min} Dashed line: η_{avg}

Case	Resolution	$\eta_{\min}/\Delta x \geq 1.5$ (New)	$\eta_{avg}/\Delta x \geq 1.5$ (Classical)
1	8192^2	√	√
2	4096^2	√	√
3	2048^2	×	√
4	1024^2	×	×

3.3 Validation of numerical criterion



Evolution of physical quantities for cases 1-4
Inset: relative error of cases 2-4 by taking case 1 as reference

Error all <1%

Error of $\langle\theta^3\rangle, \mathcal{S}$ >1%

Error all > 1%

Case	Resolu-tion	$\eta_{\min}/\Delta x \geq 1.5$ (New)	$\eta_{\text{avg}}/\Delta x \geq 1.5$ (Classical)
1	8192^2	√	√
2	4096^2	√	√
3	2048^2	×	√
4	1024^2	×	×

- Our new criterion guaranties a good resolution of 3rd order velocity gradient moment.
- The classical criterion remains useful if we only calculate the 2nd order velocity gradient moments.

4. Conclusion

4. Conclusion

- Local Kolmogorov scale: $\eta(\boldsymbol{x}) = \left(\frac{\nu(\boldsymbol{x})^2}{\frac{\partial u_i}{\partial x_j}(\boldsymbol{x})\frac{\partial u_i}{\partial x_j}(\boldsymbol{x})}\right)^{1/4}$
- The minimum scale of turbulence should be the minimum scale of local Kolmogorov scale. In compressible turbulence, it is determined by the strongest shock.
- A numerical criterion for the good resolution of 3rd order velocity gradient moment: $\eta_{\min}/\Delta x \geq 1.5$.
- The classical criterion remains useful if we only calculate the 2nd order velocity gradient moments.

Thanks a lot!

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