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Numerical methods-VIII
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[13-A-03] A positivity-preserving high-order compact finite volume method for transport eikonal equation

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A positivity-preserving high-order compact finite volume method for transport eikonal equation

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1 Introduction

It has been reported that there are numerical instabilities when solving the transport eikonal equation via high-order numerical schemes, which requires special treatments to ensure the stability of the calculation [1]. As shown in Figure 1, numerical instabilities arise on both structural and unstructured meshes.

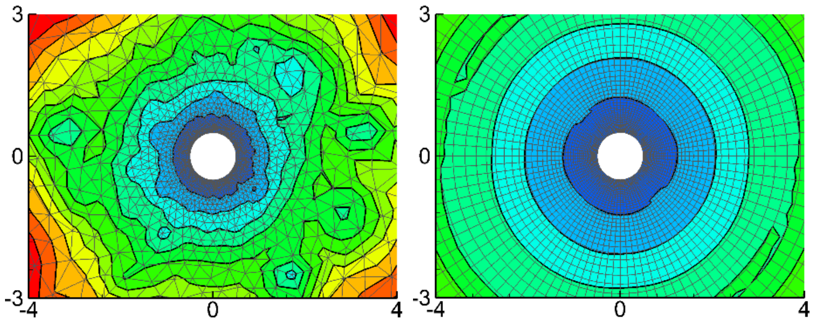


Figure 1 : Numerical instability when solving the transport eikonal equation using a fourth order compact FV scheme

In this paper, it is found that when the non-conservative convection equation is rewritten into the conservative form with additional source term, the direct application of the finite volume scheme using high order reconstruction will produce numerical instability. To solve this problem, we suggest to solve the integral form of the non-conservative convection equation directly. To account for the upwind property, a convective reconstruction (CR) technique is proposed. An artificial viscosity term is introduced to handle the weak singularity. Furthermore, a temporal and spatial decoupled positivity-preserving algorithm is used for the stable calculation on the large aspect ratio grids and non-convex geometry shapes. The numerical results show that the proposed numerical method can achieve high order accuracy and is very robust.

2 Problem Statement

The time-dependent eikonal equation takes the form

$$\frac{\partial \phi}{\partial \tau} + \vec{u} \cdot \nabla \phi = f^2. \tag{1.1}$$

where $\vec{u} = \nabla \phi$. In the FV schemes, the integral form of Eq. (1.1) takes the form

$$\frac{\partial \overline{\phi}}{\partial \tau} + \frac{1}{\overline{\Omega}_i} \iint_{\Omega_i} \vec{u} \cdot \nabla \phi d\Omega = \frac{1}{\overline{\Omega}_i} \iint_{\Omega_i} f^2 d\Omega , \tag{1.2}$$

where $\overline{\phi}_i = \frac{1}{\overline{\Omega}_i} \iint_{\Omega_i} \phi(\vec{x}, \tau) d\Omega$. Conventionally, Eq.(1.2) will be solved by its mathematically equivalent counterpart, i.e.,

$$\frac{\partial \overline{\phi}}{\partial \tau} + \frac{1}{\overline{\Omega}_i} \oint_{\partial \Omega_i} \phi \vec{u} \cdot \vec{n} dS = \frac{1}{\overline{\Omega}_i} \iint_{\Omega_i} \left(\phi \Delta \phi + f^2 \right) d\Omega . \tag{1.3}$$

It is worth to note that Eq.(1.2) is probably incompatible with Eq.(1.3) numerically, since the numerical flux of the conservative term is usually computed by a certain Riemann solver or flux splitting technique, while the source term is computed in terms of the reconstructed distribution of the dependent variable from its cell average values within the reconstruction stencil. According to our analysis on a model equation, a negative dissipation source term will be introduced when we solving Eq.(1.3) instead of Eq.(1.2). Therefore, we suggest to solve Eq.(1.2) directly to avoid introducing the numerical instability.

Generally, there are three key issues for solving Eq. (1.2): 1) how to account for the upwind effect; 2) how to handle the weak discontinuity; 3) how to keep a stable calculation on large aspect ratio grids and non-convex geometry shapes.

Firstly, we propose the CR to obtain an upwind property informed reconstructed polynomials [2]. The idea of the CR is based on the conceptual framework of variational reconstruction (VR) [3], i.e., determining the upwind property informed reconstructed polynomials via a functional extremum problem. The CR is implemented after VR. Discussions indicate that CR can inform the upwind property to the reconstructed polynomials, and the reconstruction accuracy can be preserved, as shown in Table 1. Numerical tests show that the best choice is to use CR to evaluate $\nabla\phi$ in Eq. (1.2).

Table 1 : Accuracy orders for the test cases

	Scale of grid	L_1 error	Order	L_2 error	Order	L_∞ error	Order
Grid1	1	9.780E-2		2.161E-1		5.830E-1	
Grid2	1/2	1.097E-2	3.156	4.202E-2	2.363	1.775E-1	1.715
Grid3	1/4	2.774E-4	5.306	1.380E-3	4.928	8.170E-3	4.442
Grid4	1/8	8.693E-6	4.996	5.905E-5	4.547	4.939E-4	4.048

Secondly, we introduce an artificial viscosity term to handle the weak discontinuity, i.e., we solve the following equation

$$\bar{\phi}_t + \iint_{\Omega_i} \vec{u} \cdot \nabla\phi d\Omega = \frac{1}{\bar{\Omega}_i} \iint_{\Omega_i} f^2 d\Omega + \sum_{f=1}^{Nf} \int \alpha_f \nabla\phi_f \cdot \vec{n}_f dS, \tag{1.4}$$

where

$$\alpha_f = \gamma\beta_f \left|\vec{u}_f \cdot \vec{n}_f\right| d_f$$

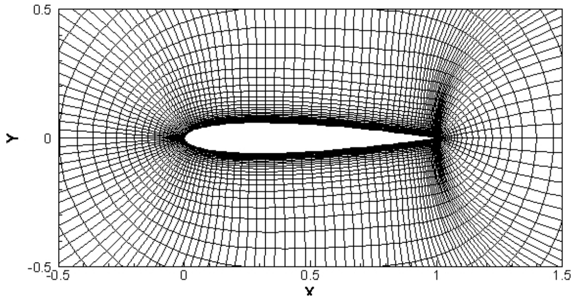
$$\beta_f = \frac{sign\left(\eta_f - \eta_{f,c}\right) + 1}{2}\eta_f$$

$$\eta_f = \left|1 - \frac{2\left|\nabla\phi_i(\vec{x}_f)\cdot\vec{n}\right|\left|\nabla\phi_j(\vec{x}_f)\cdot\vec{n}\right|}{\left|\nabla\phi_i(\vec{x}_f)\cdot\vec{n}\right|^2 + \left|\nabla\phi_j(\vec{x}_f)\cdot\vec{n}\right|^2 + \varepsilon}\right|$$

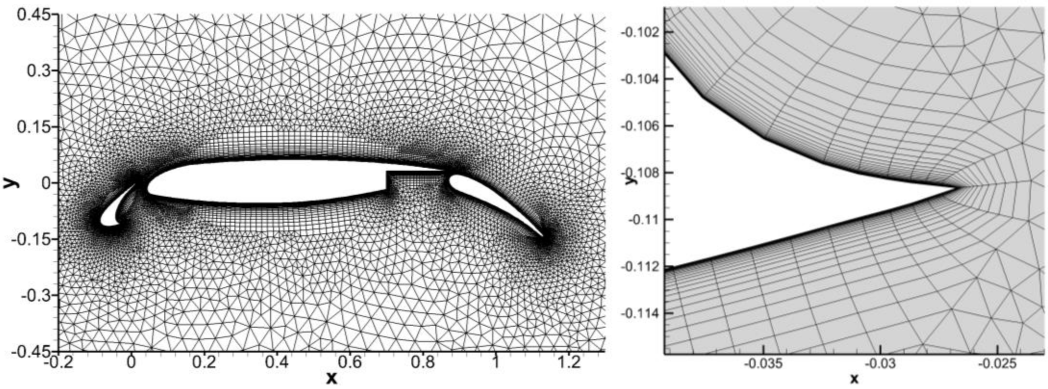
$\gamma=0.5$, and d_f is the distance between the barycenter of two cells that share the face.

Thirdly, to realize the stable calculation on the large aspect ratio grids and non-convex geometry shapes, it is critical to handle the negative numerical solutions. We adopt the temporal and spatial decoupled positivity-preserving algorithm to preserve the positivity of the cell averaged solutions and the distributions of the solution on each control volumes [4].

Figure 2 presents the computational meshes, and Figure 3 presents the results of the predicted solutions and their partial derivatives with respect to *x*, which demonstrates the effectiveness of the proposed method.

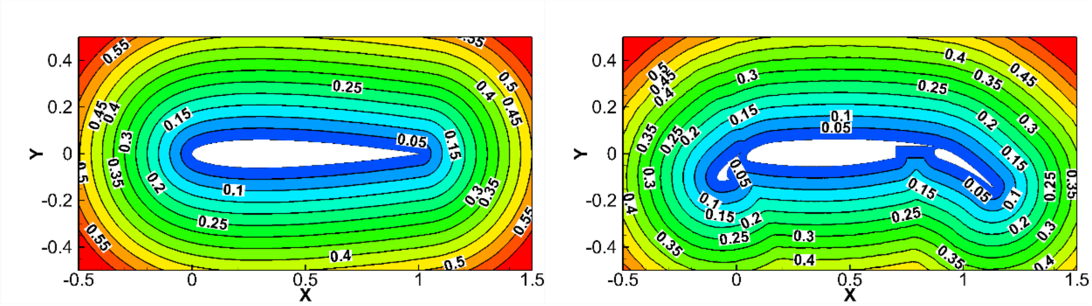


(a) NACA0012

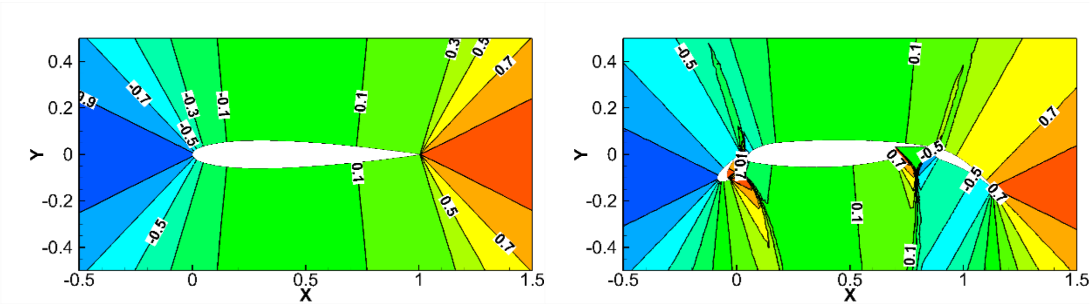


(b) 30P30N

Figure 2 : Meshes for NACA0012 and 30P30N



(a) NACA0012 (b) 30P30N



(c) NACA0012 (d) 30P30N

Figure 3 : Results of NACA0012 and 30P30N: Minimum wall distance (top), the partial derivatives of the minimum wall distance with respect to *x* (bottom)

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