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A New Approach for a Wider Class of Entropy Split Methods for Compressible Gas Dynamics and MHD

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Abstract: The high order entropy split methods of Sjögreen & Yee [1, 2] by entropy splitting of the compressible Euler (inviscid) flux derivatives for a thermally-perfect gas are based on Harten's entropy function [3, 4, 5]. Their derivation takes advantage of the homogeneity property of Euler flux, symmetrizable Euler flux derivatives and energy-norm stability in conjunction with high order classical spatial central, DRP (dispersion relation-preserving) [6, 7, 8] or Padé (compact) spatial discretizations [9] with summation-by-parts (SBP) operators [10]. Our entropy split methods have been proven entropy conserving and stable [1, 11, 12]. Our proofs do not rely on a two-point numerical flux, but rather only a linear difference operator is required to derive these methods. To extend the entropy split method for the MHD, we used the Godunov symmetrizable non-conservative MHD form [12, 13, 14]. These high order entropy split methods not only preserve certain physical properties of the chosen governing equations but are also known to either improve numerical stability, and/or minimize aliasing errors in long time integration of turbulent flow computations without the aid of added numerical dissipation. In our previous published work, extensive error norm comparison with grid refinement was performed to show the high accuracy performance of these methods. These studies also showed how well the entropy split methods conserve the entropy, momentum and mass, and preserve the kinetic energy for long time integration of the various flows [1, 2, 12, 13, 14].

The objective of the present work is to use a new approach to obtain a wider class of entropy split methods consisting of a two-point numerical flux portion and a non-conservative portion in such a way that the homogeneity property of the compressible Euler flux is not required. For high order classical spatial central, DRP (dispersion relation-preserving) or Padé (compact) spatial discretizations, this new approach can be proven to be entropy conservative with conservative spatial dsicretizations while at the same time allowing a wider class of symmetrizable inviscid flux derivatives. We also use this generalization to derive an entropy split scheme that is entropy conserving for the equations of MHD without the homogeneity property using the Godunov symmetrizable ideal MHD formulation [15].

Keywords: High Order Physical Preserving Methods, Entropy Conserving Methods, Compressible Gas Dynamics, MHD.

1 Preliminary

1.1 Euler equations of Compressible Gas Dynamics

Consider the 3D compressible Euler equations of gas dynamics as the system of conservation laws,

$$\mathbf{q}_t + \mathbf{f}_x + \mathbf{g}_y + \mathbf{h}_z = \mathbf{0} \tag{1}$$

with conserved variables

$$\mathbf{q} = (\rho \ \rho u \ \rho v \ \rho w \ e)^T,$$

where ρ denotes the density, u, v, w are the velocities in the x-, y-, and z-directions, and e denotes the total energy.

When posed on a mapped domain with coordinate mapping $x = x(\xi, \eta, \zeta)$, $y = y(\xi, \eta, \zeta)$, $z = z(\xi, \eta, \zeta)$, the equations (1) are transformed into

$$J\mathbf{q}_t + (J\xi_x\mathbf{f} + J\xi_y\mathbf{g} + J\xi_z\mathbf{h})_\xi + (J\eta_x\mathbf{f} + J\eta_y\mathbf{g} + J\eta_z\mathbf{h})_\eta + (J\zeta_x\mathbf{f} + J\zeta_y\mathbf{g} + J\zeta_z\mathbf{h})_\zeta = \mathbf{0}$$

on the domain of the unit cube, $(\xi, \eta, \zeta) \in [0, 1]^3$. Here J is the determinant of the Jacobian of the grid mapping.

The fluxes are considered in an arbitrary direction $\mathbf{k} = (k_1 \ k_2 \ k_3)$ where for mapped domains $\mathbf{k}$ are the metric derivatives, for example, for the ξ-direction fluxes, $\mathbf{k} = (J\xi_x \ J\xi_y \ J\xi_z)$. The flux of the gas dynamics equation in the direction $\mathbf{k}$ is

$$\hat{\mathbf{f}} = k_1\mathbf{f} + k_2\mathbf{g} + k_3\mathbf{h} = (\rho\hat{u}, \ \rho u\hat{u} + k_1 p, \ \rho v\hat{u} + k_2 p, \ \rho w\hat{u} + k_3 p, \ \hat{u}(e + p))^T. \tag{2}$$

Here the velocity in the $\mathbf{k}$-direction is denoted by

$$\hat{u} = k_1 u + k_2 v + k_3 w.$$

The total energy is related to the pressure p by the ideal gas law,

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2 + w^2),$$

where $\gamma > 1$ is a given constant. For simplicity the one-dimensional form of the equations are sometimes used in the description below. Writing x for the coordinate and $\mathbf{f}$ for the flux in one dimension, should be interpreted for curvilinear grids as the coordinate ξ with flux $\hat{\mathbf{f}}$ with direction vector $\mathbf{k}$ equal to the corresponding metric derivatives.

An entropy is a convex function $E = E(\mathbf{q})$ such that smooth solutions of (1) satisfy the additional scalar conservation law

$$E_t + F_x + G_y + H_z = 0, \tag{3}$$

where the entropy fluxes, F, G, and H are related to the Euler fluxes by $E_\mathbf{q}\mathbf{f}_x = F_x$, and similarly for the y- and z-directions. The entropy variables are defined by

$$\mathbf{v} = E_\mathbf{q}(\mathbf{q}),$$

which is a well-defined change of variables, $\mathbf{v} = \mathbf{v}(\mathbf{q})$, due to the convexity of $E(\mathbf{q})$.

There are mainly two entropies for the Euler equations that are used in entropy based numerical method developments: (a) Harten[3] considered the class of entropies

$$E_H = -\frac{\gamma + \alpha}{\gamma - 1}\rho(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}, \tag{4}$$

where α is a parameter. To ensure that E_H is convex, i.e., that the matrix $(E_H)_{\mathbf{q},\mathbf{q}}$ is positive definite, α is required to satisfy $\alpha > 0$ or $\alpha < -\gamma$ The corresponding x-direction entropy flux is uE. (b) The logarithmic entropy,

$$E_L = -\rho\log(p\rho^{-\gamma}), \tag{5}$$

is another commonly used entropy. Entropy conserving (EC) schemes are numerical discretizations for which the semi-discrete counterpart of (3) holds.

The high order entropy split methods by Yee et al., Sjögreen & Yee [5, 16, 1, 17, 2] split the 3D thermally perfect gas dynamics Euler equations into conservative and non-conservative parts before discretization as

$$\mathbf{q}_t + \frac{\beta}{\beta+1}(\mathbf{f}_x^{(x)} + \mathbf{f}_y^{(y)} + \mathbf{f}_z^{(z)}) + \frac{1}{\beta+1}(A^{(x)}\mathbf{v}_x + A^{(y)}\mathbf{v}_y + A^{(z)}\mathbf{v}_z) = \mathbf{0}. \tag{6}$$

The matrix in the x-direction is defined by

$$A^{(x)} = \mathbf{f}_{\mathbf{v}}^{(x)} \tag{7}$$

with $\beta \neq -1$ and for physical relevant solutions, we must choose positive split parameter $\beta > 0$. Similarly for the other coordinate directions.

In the original studies by Harten and Gerritsen & Olsson [3, 4], they allowed negative β. Yee et al., Vinokur & Yee and Sjögreen et al. [5, 18, 19] re-examined the split form of the Euler flux derivative by [3, 4] to pick the physically relevant branch of the split parameter β. They extended the entropy split form to include thermally-perfect gas for moving curvilinear grids. In addition, they performed a detailed study by applying a high order central scheme on the entropy splitting form to the Euler flux derivatives and comparing with the un-split form, including a 3D channel turbulence flow [20]. It was shown that the split form of the Euler flux derivatives is more stable for longer time integration without the need of added numerical dissipation. In [11], DRP (dispersion relation-preserving) finite discretizations [8, 6, 7, 21] were also applied to the entropy split form of the Euler flux derivatives and resulted in a similar gain in numerical stability.

1.2 The Original High Order Entropy Split Methods using Linear Difference Operators [5, 16, 1, 17, 2]

In Sjögreen & Yee [1, 17, 2] the entropy split method is proven to be entropy conservative and stable for a thermally-perfect gas. The various high order methods resulting from applying classical spatial central, DRP and Padé (compact) methods to the split form of the Euler flux derivative are entropy conservative.

The entropy split method is different from most standard methods in two ways. Firstly, it does not rely on a two-point numerical flux but rather uses a linear difference operator to define the method. Secondly, without further development in Sjögreen & Yee [2], the original entropy split method conserves the entropy (4) but the method itself is not in conservative form.

For simplicity, the semi-discrete entropy split approximation of the 1D version of (1) is

$$\frac{d}{dt}\mathbf{q}_j + \frac{\beta}{\beta+1}D\mathbf{f}_j + \frac{1}{\beta+1}(\mathbf{f}_{\mathbf{v}})_j D\mathbf{v}_j = 0, \quad j = 1, \ldots, N, \tag{8}$$

where D is a linear finite difference operator, and $\beta > 0$ is a parameter related to α in (4) by $\beta = (\alpha + \gamma)/(1-\gamma)$.

The flux Jacobian matrix with respect to the entropy variables, $\mathbf{f}_{\mathbf{v}}$, is symmetric. In the most common case D is a standard high-order SBP centered difference operator, but other operators are possible. For example, D could be a bandwidth optimized operator with SBP closure such as developed in [8, 6, 7, 21].

If the difference operator D has the SBP property, then the entropy conserving property

$$\Delta x \frac{d}{dt}\sum_{j=1}^{N} \omega_j E(t) - F_1 + F_N = 0 \tag{9}$$

can be proved for (8); see [1].

The entropy splitting described above as a function of the positive $\beta > 0$ parameter weights the non-conservative portion of the flux derivative by $\frac{1}{1+\beta}$. This means that the range $\beta > 0$ corresponds to a weight of non-conservative portion that is less than 1, whereas $\beta < 0$ leads, unphysically, corresponds to a weight that is greater than 1. For long time integration of certain smooth flows, our previous studies indicated that $\beta = 1$ (equal conservative portion and nonconservative portion) can help with stability without added numerical dissipation using high order central spatial discretization. For shock-free turbulence, $\beta = 1$ or 2, similar behavior was observed. For turbulence with shocklets, $\beta = 2$ or higher is needed for conservative property. For turbulence with shocks, new forms of the entropy split method, including the derivation of a high order conservative numerical flux for the non-conservative portion of the entropy splitting of the Euler flux derivatives in conjunction with their nonlinear filter approach were proposed in Sjögreen & Yee [2].

It is important to point out that for $\beta = 1$, the entropy split method using central spatial scheme becomes the standard Ducros et al. splitting [22], and can be easily rewritten into a conservation form of arbitrary order using central spatial discretizations."

These high order entropy split methods not only preserve certain physical properties of the chosen governing equations but are also known to either improve numerical stability, and/or minimize aliasing errors in long time integration of turbulent flow computations without the aid of added numerical dissipation. In our previous published work, extensive error norm comparison with grid refinement was performed to show the high accuracy performance of these methods. These studies also showed how well the entropy split methods conserve the entropy, momentum and mass, and preserve the kinetic energy for long time integration of the various flows [1, 2, 12, 13, 14]. The shortcoming is that these methods are not applicable to the Euler flux without the homogeneity property.

Our objective here is to present a new formulation to relax the homogeneity requirement to obtain entropy split methods that are entropy conserving. We next describe how the entropy split method can be generalized to cases where homogeneity does not hold but are entropy conserving. Furthermore, we use this generalization to derive an entropy split method that is entropy conserving for the equations of MHD.

2 New Formulation for High Order Entropy Split Methods Using Two-Point Numerical Flux Approach

Consider the split form of a second order accurate semi-discrete conservation law

$$\Delta x \frac{d\mathbf{q}_j}{dt} + \omega(\mathbf{h}_{j+1/2} - \mathbf{h}_{j-1/2}) + (1-\omega)A_j \frac{\mathbf{v}_{j+1} - \mathbf{v}_{j-1}}{2} = \mathbf{0}, \quad j = 1, \ldots, N \tag{10}$$

where $\mathbf{h}_{j+1/2}$ is a numerical flux function consistent up to second order with the flux $\mathbf{f}$ of the continuous problem, and as above $A = \partial \mathbf{f}/\partial \mathbf{v}$. The constant weight ω, with $0 \le \omega \le 1$, is the fraction of the conservative part of the discretization. We have the following result.

Theorem 1. *Assume that the numerical fluxes $\mathbf{h}_{j+1/2}$ of (10) are determined such that*

$$\omega(\mathbf{v}_{j+1} - \mathbf{v}_j)^T \mathbf{h}_{j+1/2} = (1-\omega)\frac{1}{2}(\mathbf{v}_{j+1}^T A_{j+1} + \mathbf{v}_j^T A_j)(\mathbf{v}_{j+1} - \mathbf{v}_j) + \psi_{j+1} - \psi_j, \tag{11}$$

where $\psi = \omega \mathbf{v}^T \mathbf{f} - F$. Then the discretization (10) implies the entropy conservation law

$$\Delta x \frac{d}{dt} E_j(t) + H_{j+1/2} - H_{j-1/2} = 0, \tag{12}$$

where the discrete entropy fluxes are given by

$$H_{j+1/2} = \frac{\omega}{2}(\mathbf{v}_{j+1} + \mathbf{v}_j)^T \mathbf{h}_{j+1/2} - \frac{1-\omega}{4}(\mathbf{v}_{j+1}^T A_{j+1} - \mathbf{v}_j^T A_j)(\mathbf{v}_{j+1} - \mathbf{v}_j) - \frac{1}{2}(\psi_{j+1} + \psi_j). \tag{13}$$

Proof: We multiply (10) by $\mathbf{v}_j^T$. This gives

$$\Delta x \frac{dE_j}{dt} + \omega \mathbf{v}_j^T (\mathbf{h}_{j+1/2} - \mathbf{h}_{j-1/2}) + (1-\omega)\mathbf{v}_j^T A_j \frac{\mathbf{v}_{j+1} - \mathbf{v}_{j-1}}{2} = \mathbf{0}.$$

The flux difference term is rewritten

$$\mathbf{v}_j^T(\mathbf{h}_{j+1/2}-\mathbf{h}_{j-1/2}) = \frac{1}{2}(\mathbf{v}_{j+1}+\mathbf{v}_j)^T\mathbf{h}_{j+1/2} - \frac{1}{2}(\mathbf{v}_j+\mathbf{v}_{j-1})^T\mathbf{h}_{j-1/2} - \frac{1}{2}(\mathbf{v}_{j+1}-\mathbf{v}_j)^T\mathbf{h}_{j+1/2} - \frac{1}{2}(\mathbf{v}_j-\mathbf{v}_{j-1})^T\mathbf{h}_{j-1/2}. \quad (14)$$

The non-conservative term is rewritten

$$\mathbf{v}_j^T A_j(\mathbf{v}_{j+1}-\mathbf{v}_{j-1}) = \frac{1}{2}(\mathbf{v}_{j+1}^T A_{j+1}+\mathbf{v}_j^T A_j)(\mathbf{v}_{j+1}-\mathbf{v}_j) + \frac{1}{2}(\mathbf{v}_j^T A_j+\mathbf{v}_{j-1}^T A_{j-1})(\mathbf{v}_j-\mathbf{v}_{j-1}) - \frac{1}{2}(\mathbf{v}_{j+1}^T A_{j+1}-\mathbf{v}_j^T A_j)(\mathbf{v}_{j+1}-\mathbf{v}_j) + \frac{1}{2}(\mathbf{v}_j^T A_j-\mathbf{v}_{j-1}^T A_{j-1})(\mathbf{v}_j-\mathbf{v}_{j-1}). \quad (15)$$

Adding together and collecting terms that are differences gives

$$\Delta x\frac{dE_j}{dt} + \Delta_+(\omega\frac{1}{2}(\mathbf{v}_j+\mathbf{v}_{j-1})^T\mathbf{h}_{j-1/2} - (1-\omega)\frac{1}{4}(\mathbf{v}_j^T A_j-\mathbf{v}_{j-1}^T A_{j-1})(\mathbf{v}_j-\mathbf{v}_{j-1})) - \omega\frac{1}{2}(\mathbf{v}_{j+1}-\mathbf{v}_j)^T\mathbf{h}_{j+1/2} + (1-\omega)\frac{1}{4}(\mathbf{v}_{j+1}^T A_{j+1}+\mathbf{v}_j^T A_j)(\mathbf{v}_{j+1}-\mathbf{v}_j) - \omega\frac{1}{2}(\mathbf{v}_j-\mathbf{v}_{j-1})^T\mathbf{h}_{j-1/2} + (1-\omega)\frac{1}{4}(\mathbf{v}_j^T A_j+\mathbf{v}_{j-1}^T A_{j-1})(\mathbf{v}_j-\mathbf{v}_{j-1}) = 0, \quad (16)$$

where the first line is of conservation form. The forward difference is defined by $\Delta_+(u_j) = u_{j+1} - u_j$. The second and third lines can, by (11), be replaced by $-(\psi_{j+1}-\psi_j)/2$ and $-(\psi_j-\psi_{j-1})/2$ respectively. We obtain

$$\Delta x\frac{dE_j}{dt} + \Delta_+(\omega\frac{1}{2}(\mathbf{v}_j+\mathbf{v}_{j-1})^T\mathbf{h}_{j-1/2} - (1-\omega)\frac{1}{4}(\mathbf{v}_j^T A_j-\mathbf{v}_{j-1}^T A_{j-1})(\mathbf{v}_j-\mathbf{v}_{j-1})) - \frac{1}{2}(\psi_{j+1}-\psi_j) - \frac{1}{2}(\psi_j-\psi_{j-1}) = 0. \quad (17)$$

Entropy conservation with the numerical entropy flux (13) follows from (17) by rewriting the ψ-terms as

$$-\frac{1}{2}(\psi_{j+1}+\psi_j) + \frac{1}{2}(\psi_j+\psi_{j-1}) = -\frac{1}{2}\Delta_+(\psi_j+\psi_{j-1}).$$

This shows (12), and thereby proves the theorem.

According to Theorem 1, all that is needed in order to define an entropy split method of form (10) that conserves entropy is to determine the numerical flux function, $\mathbf{h}_{j+1/2}$, so that it satisfies (11). As an example, this is next carried out for the equations of gas dynamics, using the flux (2) and the entropy (4). For the gas dynamics equations, the entropy split form (8) is known to conserve entropy. This example will demonstrate that (8) can be derived by using Theorem 1.

The entropy variables corresponding to the Harten entropy functions (4) are

$$\mathbf{v} = \frac{\rho}{p}s^{\frac{1}{\alpha+\gamma}}(-\frac{\alpha}{\gamma-1}\frac{p}{\rho} - \frac{1}{2}|\mathbf{u}|^2,\ u,\ v,\ w,\ -1)^T. \quad (18)$$

Multiplying together (18) and the inviscid flux gives

$$\mathbf{v}^T\mathbf{f} = -\frac{\alpha+1}{\gamma-1}\rho\hat{u}s^{\frac{1}{\alpha+\gamma}},$$

so that

$$\psi = \omega\mathbf{v}^T\mathbf{f} - F = -\omega\frac{\alpha+1}{\gamma-1}\rho\hat{u}s^{\frac{1}{\alpha+\gamma}} - \hat{u}E = \omega\frac{\alpha+1}{\alpha+\gamma}\hat{u}E - \hat{u}E,$$

where the second equality follows from the entropy definition (4). If we take $\omega = \frac{\alpha+\gamma}{\alpha+1} = \beta/(\beta+1)$, then $\psi = 0$. Furthermore, by using $\psi = 0$ and $\mathbf{v}^T A = \beta \mathbf{f}$ (which can be shown by directly evaluating the product $\mathbf{v}^T A$) (11) becomes

$$(\mathbf{v}_{j+1} - \mathbf{v}_j)^T \mathbf{h}_{j+1/2} = \frac{1}{2}(\mathbf{v}_{j+1} - \mathbf{v}_j)^T(\mathbf{f}_{j+1} + \mathbf{f}_j)$$

which clearly can be satisfied by defining $\mathbf{h}_{j+1/2}$ as the standard second-order accurate centered flux, i.e., $\mathbf{h}_{j+1/2} = (\mathbf{f}_j + \mathbf{f}_{j+1})/2$. In this sense the entropy split method follows from (11) without explicit use of the homogeneity. A byproduct from this way of deriving the entropy split method is that, thanks to Theorem 1, the method is consistent with the local entropy conservation law (12) with numerical entropy flux

$$H_{j+1/2} = \frac{\beta}{(\beta+1)}\frac{1}{4}((\mathbf{v}_{j+1} + \mathbf{v}_j)^T(\mathbf{f}_{j+1} + \mathbf{f}_j) - (\mathbf{v}_{j+1} - \mathbf{v}_j)^T(\mathbf{f}_{j+1} - \mathbf{f}_j)).$$

This local entropy conservation is not obtained from the standard SBP analysis leading to (9).

Hence, for fluxes that are not homogeneous, it would be possible to derive an entropy split method by determining numerical fluxes that satisfy (11). Such a method would be more complicated, since the conservative part of the split formula would not necessarily be a standard centered approximation.

The above shows the proofs used the second-order classical spatial center methods. In the same procedure as indicated in Sjögreen & Yee, and Sjögreen et al. [17, 1, 11, 12], it is straightforward to extend the new formulation to higher than second-order classical central, DRP and Padé spatial discretizations. We next investigate the extension the two-point numerical flux entropy split method that is entropy conserving to the equations of ideal MHD with fluxes that do not have the homogeneity property.

3 Extension of the New Formulation for the Equations of MHD

The equations of MHD do not have homogeneous fluxes. Furthermore, the flux Jacobians with respect to the entropy variables are not symmetric. Hence, the standard technique used for gas dynamics to prove that (8) conserves entropy does not carry over to the corresponding entropy split form of the equations of MHD. No entropy conserving entropy split form were previously known for the equations of MHD. This section shows how the extension of Theorem 1 to MHD can be used to define an entropy conserving entropy split approximation. Unlike the gas dynamics case, the numerical flux of the conservative part $\mathbf{h}_{j+1/2}$ of the MHD entropy split approximation is not the standard centered flux.

Consider the ideal MHD system,

$$\mathbf{q}_t + \mathbf{f}_x + \mathbf{g}_y + \mathbf{h}_z + \mathbf{e}(\nabla\cdot \mathbf{B}) = \mathbf{0}$$

with conserved variables

$$\mathbf{q} = (\rho,\ \rho u_1,\ \rho u_2,\ \rho u_3,\ e,\ B_1,\ B_2,\ B_3)^T,$$

and introduce the notation $\mathbf{u} = (u_1\ u_2\ u_3)^T$, $\mathbf{B} = (B_1\ B_2\ B_3)^T$, $|\mathbf{u}|^2 = u_1^2+u_2^2+u_3^2$, and $|\mathbf{B}|^2 = B_1^2+B_2^2+B_3^2$. The flux in direction $(k_1,\ k_2,\ k_3)$ is

$$\mathbf{f} = \begin{pmatrix} \rho\hat{u} \\ \rho\hat{u}u_1 + k_1(p+\frac{1}{2}|\mathbf{B}|^2) - \hat{B}B_1 \\ \rho\hat{u}u_2 + k_2(p+\frac{1}{2}|\mathbf{B}|^2) - \hat{B}B_2 \\ \rho\hat{u}u_3 + k_3(p+\frac{1}{2}|\mathbf{B}|^2) - \hat{B}B_3 \\ \hat{u}(e+p+\frac{1}{2}|\mathbf{B}|^2) - \hat{B}\mathbf{u}^T\mathbf{B} \\ \hat{u}B_1 - \hat{B}u_1 \\ \hat{u}B_2 - \hat{B}u_2 \\ \hat{u}B_3 - \hat{B}u_3 \end{pmatrix}, \tag{19}$$

where $\hat{u} = k_1 u_1 + k_2 u_2 + k_3 u_3$ and $\hat{B} = k_1 B_1 + k_2 B_2 + k_3 B_3$. The "source term" vector is

$$\mathbf{e} = (0,\ B_1,\ B_2,\ B_3,\ \mathbf{u}^T\mathbf{B},\ u_1,\ u_2,\ u_3)^T,$$

which was used in [15]. But other choices are possible. The fluxes do not have the homogeneity property. The total energy is related to the pressure p by the constitutive law,

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{B}|^2.$$

Here we will use the Harten entropies (4). It is straightforward to verify that $E_{\mathbf{q},\mathbf{q}} > 0$ for $\alpha < -\gamma$ or $\alpha > 0$ also for the equations of MHD. Similarly, the entropy flux is $F = \hat{u}E$, just as in the gas dynamics case. The entropy variables $\mathbf{v} = E_\mathbf{q}$ are found by differentiation to be

$$\mathbf{v} = \frac{\rho}{p} s^{\frac{1}{\alpha+\gamma}} \left(-\frac{\alpha}{\gamma - 1}\frac{p}{\rho} - \frac{1}{2}|\mathbf{u}|^2,\ u_1,\ u_2,\ u_3,\ -1,\ B_1,\ B_2,\ B_3\right)^T. \tag{20}$$

Consider the 1D split discretization of MHD,

$$\Delta x \frac{d\mathbf{q}_j}{dt} + \omega(\mathbf{h}_{j+1/2} - \mathbf{h}_{j-1/2}) + (1-\omega)A_j \frac{\mathbf{v}_{j+1} - \mathbf{v}_{j-1}}{2} + \omega \mathbf{e}_j \frac{\hat{B}_{j+1} - \hat{B}_{j-1}}{2} = \mathbf{0}, \tag{21}$$

for $j = 1, \ldots, N$, where $\mathbf{h}_{j+1/2}$ is a numerical flux to be determined to make (21) conserve entropy. The symmetric matrix A is given by

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{v}} + C,$$

where C is the matrix obtained by rewriting $\mathbf{e}\hat{B}_x = C\mathbf{v}_x$. A simple modification of Theorem 1 to include the MHD "source term" gives

Theorem 2. *Assume that the numerical fluxes $\mathbf{h}_{j+1/2}$ of (21) satisfy*

$$\begin{aligned}\omega(\mathbf{v}_{j+1} - \mathbf{v}_j)^T \mathbf{h}_{j+1/2} = (1-\omega)\frac{1}{2}(\mathbf{v}_{j+1}^T A_{j+1} + \mathbf{v}_j^T A_j)(\mathbf{v}_{j+1} - \mathbf{v}_j) \\ + \frac{\omega}{2}(\mathbf{v}_{j+1}^T \mathbf{e}_{j+1} + \mathbf{v}_j^T \mathbf{e}_j)(\hat{B}_{j+1} - \hat{B}_j) + \psi_{j+1} - \psi_j,\end{aligned} \tag{22}$$

where $\psi = \omega \mathbf{v}^T \mathbf{f} - F$. Then the discretization (21) implies the entropy conservation law

$$\Delta x \frac{E_j}{dt} + H_{j+1/2} - H_{j-1/2} = 0,$$

where the discrete entropy fluxes are given by

$$\begin{aligned}H_{j+1/2} = \frac{\omega}{2}(\mathbf{v}_{j+1} + \mathbf{v}_j)^T \mathbf{h}_{j+1/2} - \frac{1-\omega}{4}(\mathbf{v}_{j+1}^T A_{j+1} - \mathbf{v}_j^T A_j)(\mathbf{v}_{j+1} - \mathbf{v}_j) \\ - \frac{\omega}{4}(\mathbf{v}_{j+1}^T \mathbf{e}_{j+1} - \mathbf{v}_j^T \mathbf{e}_j)(\hat{B}_{j+1} - \hat{B}_j) - \frac{1}{2}(\psi_{j+1} + \psi_j).\end{aligned} \tag{23}$$

To see what this means for the equations of MHD, it is straightforward to evaluate

$$\mathbf{v}^T \mathbf{f} = -\frac{\alpha + 1}{\gamma - 1}\rho \hat{u} s^{\frac{1}{\alpha+\gamma}} + \frac{\rho}{p} s^{\frac{1}{\alpha+\gamma}} \left(\frac{1}{2}\hat{u}|\mathbf{B}|^2 - \mathbf{u}^T \mathbf{B}\hat{B}\right).$$

The weighted entropy flux potential becomes

$$\psi = \omega \mathbf{v}^T \mathbf{f} - F = \left(-\omega\frac{\alpha + 1}{\gamma - 1} + \frac{\alpha + \gamma}{\gamma - 1}\right)\rho \hat{u} s^{\frac{1}{\alpha+\gamma}} + \omega \frac{\rho}{p} s^{\frac{1}{\alpha+\gamma}} \left(\frac{1}{2}\hat{u}|\mathbf{B}|^2 - \hat{B}\mathbf{u}^T \mathbf{B}\right).$$

To apply (22) the quantity $\mathbf{v}^T A$ is required. A straightforward but lengthy evaluation gives

$$\mathbf{v}^T A = -\frac{\alpha+\gamma}{\gamma-1}(\rho\hat{u},\ \rho\hat{u}u_1 + k_1 p,\ \rho\hat{u}u_2 + k_2 p,\ \rho\hat{u}u_3 + k_3 p,\ \hat{u}(e+p-\frac{1}{2}|\mathbf{B}|^2),\ 0,\ 0,\ 0). \tag{24}$$

This is essentially the gas dynamics part of the flux function. Split the MHD flux (19) into a gas dynamics part and a magnetic part, $\mathbf{f} = \mathbf{f}^G + \mathbf{f}^M$, with

$$\mathbf{f}^G = \begin{pmatrix} \rho\hat{u} \\ \rho\hat{u}u_1 + k_1 p \\ \rho\hat{u}u_2 + k_2 p \\ \rho\hat{u}u_3 + k_3 p \\ \hat{u}(e+p-\frac{1}{2}|\mathbf{B}|^2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{f}^M = \begin{pmatrix} 0 \\ k_1\frac{1}{2}|\mathbf{B}|^2 - \hat{B}B_1 \\ k_2\frac{1}{2}|\mathbf{B}|^2 - \hat{B}B_2 \\ k_3\frac{1}{2}|\mathbf{B}|^2 - \hat{B}B_3 \\ \hat{u}|\mathbf{B}|^2 - \hat{B}\mathbf{u}^T\mathbf{B} \\ \hat{u}B_1 - \hat{B}u_1 \\ \hat{u}B_2 - \hat{B}u_2 \\ \hat{u}B_3 - \hat{B}u_3 \end{pmatrix}.$$

Define $\beta = -(\alpha+\gamma)/(\gamma-1)$. It is seen that $\mathbf{v}^T A = \beta\mathbf{f}^G$. Furthermore, define

$$\omega = (\alpha+\gamma)/(\alpha+1) = \beta/(\beta+1).$$

We obtain

$$\psi = \omega\frac{\rho}{p}s^{\frac{1}{\alpha+\gamma}}\left(\frac{1}{2}\hat{u}|\mathbf{B}|^2 - \hat{B}\mathbf{u}^T\mathbf{B}\right). \tag{25}$$

Because $(1-\omega)\beta = \omega$, equation (22) can be rewritten

$$(\mathbf{v}_{j+1}-\mathbf{v}_j)^T\mathbf{h}_{j+1/2} = \frac{1}{2}(\mathbf{f}^G_{j+1}+\mathbf{f}^G_j)(\mathbf{v}_{j+1}-\mathbf{v}_j) + \frac{1}{2}(\mathbf{v}^T_{j+1}\mathbf{e}_{j+1}+\mathbf{v}^T_j\mathbf{e}_j)(\hat{B}_{j+1}-\hat{B}_j) + \psi^M_{j+1} - \psi^M_j, \tag{26}$$

with ψ^M given by

$$\psi^M = \frac{\rho}{p}s^{\frac{1}{\alpha+\gamma}}\left(\frac{1}{2}\hat{u}|\mathbf{B}|^2 - \hat{B}\mathbf{u}^T\mathbf{B}\right). \tag{27}$$

The numerical flux of the entropy split method is selected as

$$\mathbf{h}_{j+1/2} = \frac{1}{2}(\mathbf{f}^G_{j+1}+\mathbf{f}^G_j) + \mathbf{h}^M_{j+1/2},$$

where the magnetic part of the numerical flux is required to satisfy

$$(\mathbf{v}_{j+1}-\mathbf{v}_j)^T\mathbf{h}^M_{j+1/2} = \frac{1}{2}(\mathbf{v}^T_{j+1}\mathbf{e}_{j+1}+\mathbf{v}^T_j\mathbf{e}_j)(\hat{B}_{j+1}-\hat{B}_j) + \psi^M_{j+1} - \psi^M_j. \tag{28}$$

3.1 Derivation of a Split Form Entropy Conserving Method

Next, we derive the magnetic part of the numerical flux function from (28).

Introducing the parameter vector

$$\mathbf{z} = (\frac{\rho}{p}s^{\frac{1}{\alpha+\gamma}},\ u_1,\ u_2,\ u_3,\ p,\ B_1,\ B_2,\ B_3),$$

leads to the following expressions for the entropy variables and entropy flux potential in terms of the components of $\mathbf{z}$,

$$\begin{aligned} v_1 &= -\frac{1}{2}z_1(z_2^2+z_3^2+z_4^2) - \frac{\alpha}{\gamma-1}z_1^{-\frac{\gamma}{\alpha}}z_5^{-\frac{\gamma-1}{\alpha}} \\ v_2 &= z_1z_2 \quad v_3 = z_1z_3 \quad v_4 = z_1z_4 \\ v_5 &= -z_1 \\ v_6 &= z_1z_6 \quad v_7 = z_1z_7 \quad v_8 = z_1z_8 \end{aligned} \tag{29}$$

where $\hat{z} = k_1z_2 + k_2z_3 + k_3z_4$ and $\tilde{z} = k_1z_6 + k_2z_7 + k_3z_8$. Furthermore,

$$\mathbf{e}^T\mathbf{v} = z_1(z_2z_6 + z_3z_7 + z_4z_8).$$

We will determine a magnetic numerical flux function, $\mathbf{h}^M_{j+1/2} = \mathbf{h}(\mathbf{u}_{j+1}, \mathbf{u_j})$, so that it statisfies (28), expressed as

$$(\mathbf{h}^M_{j+1/2})^T\Delta\mathbf{v} = \{\mathbf{v}^T\mathbf{e}\}\Delta\hat{B} + \Delta\psi^M, \tag{30}$$

where the difference and average are denoted by

$$\Delta u = (u_{j+1} - u_j) \quad \text{and} \quad \{u\} = (u_{j+1} + u_j)/2,$$

respectively.

Expressed in the intermediate variables, the right hand side of (30) becomes

$$\{z_1(z_2z_6 + z_3z_7 + z_4z_8)\}\Delta\tilde{z} + \Delta\left(\frac{1}{2}\hat{z}z_1(z_6^2 + z_7^2 + z_8^2)\right) - \Delta\left(\tilde{z}z_1(z_2z_6 + z_3z_7 + z_4z_8)\right),$$

which we expand as

$$\begin{aligned} &\frac{1}{2}\{\hat{z}\}\{z_6^2 + z_7^2 + z_8^2\}\Delta z_1 + \{\hat{z}z_1\}(\{z_6\}\Delta z_6 + \{z_7\}\Delta z_7 + \{z_8\}\Delta z_8) \\ &\quad + \frac{1}{2}\{z_1\}\{z_6^2 + z_7^2 + z_8^2\}\Delta\hat{z} - \{\tilde{z}\}(\{z_2\}\{z_6\} + \{z_3\}\{z_7\} + \{z_4\}\{z_8\})\Delta z_1 \\ &\quad - \{\tilde{z}\}\{z_1\}(\{z_6\}\Delta z_2 + \{z_7\}\Delta z_3 + \{z_8\}\Delta z_4) \\ &\quad - \{\tilde{z}\}(\{z_1z_2\}\Delta z_6 + \{z_1z_3\}\Delta z_7 + \{z_1z_4\}\Delta z_8). \end{aligned} \tag{31}$$

For the left hand side of (30) the entropy variables are written out in terms of $\mathbf{z}$ to obtain

$$\begin{aligned} &h^{(2)}\{z_1\}\Delta z_2 + h^{(3)}\{z_1\}\Delta z_3 + h^{(4)}\{z_1\}\Delta z_4 \\ &\quad + \left(-h^{(5)} + \{z_2\}h^{(2)} + \{z_3\}h^{(3)} + \{z_4\}h^{(4)} + \{z_6\}h^{(6)} + \{z_7\}h^{(7)} + \{z_8\}h^{(8)}\right)\Delta z_1 \\ &\quad + h^{(6)}\{z_1\}\Delta z_6 + h^{(7)}\{z_1\}\Delta z_7 + h^{(8)}\{z_1\}\Delta z_8, \end{aligned} \tag{32}$$

where $h^{(j)}$, $j = 1, \ldots, 8$, denote the components of the magnetic numerical flux, $\mathbf{h}^M$. By equating Δz_j terms, the magnetic part of the numerical flux is found to be

$$\mathbf{h}^M = \begin{pmatrix} 0 \\ \frac{1}{2}k_1\{z_6^2 + z_7^2 + z_8^2\} - \{\tilde{z}\}\{z_6\} \\ \frac{1}{2}k_2\{z_6^2 + z_7^2 + z_8^2\} - \{\tilde{z}\}\{z_7\} \\ \frac{1}{2}k_3\{z_6^2 + z_7^2 + z_8^2\} - \{\tilde{z}\}\{z_8\} \\ \frac{\{z_1\hat{z}\}}{\{z_1\}}(\{z_6\}^2 + \{z_7\}^2 + \{z_8\}^2) - \{\tilde{z}\}(\frac{\{z_1z_2\}}{\{z_1\}}\{z_6\} + \frac{\{z_1z_3\}}{\{z_1\}}\{z_7\} + \frac{\{z_1z_4\}}{\{z_1\}}\{z_8\}) \\ \frac{\{z_1\hat{z}\}}{\{z_1\}}\{z_6\} - \{\tilde{z}\}\frac{\{z_1z_2\}}{\{z_1\}} \\ \frac{\{z_1\hat{z}\}}{\{z_1\}}\{z_7\} - \{\tilde{z}\}\frac{\{z_1z_3\}}{\{z_1\}} \\ \frac{\{z_1\hat{z}\}}{\{z_1\}}\{z_8\} - \{\tilde{z}\}\frac{\{z_1z_4\}}{\{z_1\}} \end{pmatrix}.$$

In terms of the original variables, the flux is

$$\mathbf{h}^M = \begin{pmatrix} 0 \\ \frac{1}{2}k_1\{B_1^2 + B_2^2 + B_3^2\} - \{\hat{B}\}\{B_1\} \\ \frac{1}{2}k_2\{B_1^2 + B_2^2 + B_3^2\} - \{\hat{B}\}\{B_2\} \\ \frac{1}{2}k_3\{B_1^2 + B_2^2 + B_3^2\} - \{\hat{B}\}\{B_3\} \\ \frac{\{z_1\hat{u}\}}{\{z_1\}}(\{B_1\}^2 + \{B_2\}^2 + \{B_3\}^2) - \{\hat{B}\}(\frac{\{z_1u_1\}}{\{z_1\}}\{B_1\} + \frac{\{z_1u_2\}}{\{z_1\}}\{B_2\} + \frac{\{z_1u_3\}}{\{z_1\}}\{B_3\}) \\ \frac{\{z_1\hat{u}\}}{\{z_1\}}\{B_1\} - \{\hat{B}\}\frac{\{z_1u_1\}}{\{z_1\}} \\ \frac{\{z_1\hat{u}\}}{\{z_1\}}\{B_2\} - \{\hat{B}\}\frac{\{z_1u_2\}}{\{z_1\}} \\ \frac{\{z_1\hat{u}\}}{\{z_1\}}\{B_3\} - \{\hat{B}\}\frac{\{z_1u_3\}}{\{z_1\}} \end{pmatrix},$$

with $z_1 = \frac{\rho}{p}s^{\frac{1}{\alpha+\gamma}}$.

Hence, the entropy split scheme for MHD (21) with numerical flux function

$$\mathbf{h} = \{\mathbf{f}^G\} + \mathbf{h}^M, \tag{33}$$

and with $\omega = \beta/(\beta + 1)$ has mathematically strict entropy conservation. Note that only the gas dynamics part, $\mathbf{f}^G$ of the numerical flux (33) is a standard centered flux. The magnetic field part of (33) is of a form similar to a Tadmor-type entropy conserving numerical flux.

4 Conclusion and Future Work

A new approach to obtain a wider class of entropy split methods consisting of a two-point numerical flux portion and a non-conservative portion has been found that does not require the Euler flux with the homogeneity property. For high order classical spatial central, DRP (dispersion relation-preserving) or Padé (compact) spatial discretizations, this new approach can be proven to be entropy conservative with conservative spatial-dsicretizations while at the same time allowing a wider class of symmetrizable inviscid flux derivatives, including the Godunov symmetriazable ideal MHD [15] with fluxes that do not have the homogeneity property. Implementation of the new approach into our 3D research code for representative DNS & LES test case is forthcoming.

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