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Discrete Exterior Calculus for the Simulation of Forced 2D Turbulent Flows on a Sphere

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Abstract: Lastly, numerical simulation of solar convection is an inevitable tool to understand complex dynamics of convective plasma on the Sun. Previous studies showed that linear models are able to predict high latitude inertial and equatorial Rossby waves satisfactorily. However, they are inadequate in predicting the critical latitude. The focus of this paper is to handle this issue by taking into account the non linear term in the Navier-Stokes momentum equations and adding a specific forcing term to it. Numerical simulation of a forced 2D turbulent flow in a rotating sphere is then performed by using Discrete Exterior Calculus (DEC). The choice of DEC is motivated by the fact that resulting discretization has certain desirable conservation and mimetic properties that are appropriate for investigating flows dominated by long-lived coherent structures.

Keywords: 2D Turbulence, Inertial waves, Discrete Exterior Calculus.

1 Introduction

Recently, helioseismic observations performed by Gizon et al. resulted in the detection of inertial modes in the sun's surface [1]. Due to their low frequencies and amplitudes, the observation process is time consuming given the length of one period of 27 days. In order to reduce the experimental burden, numerical approaches that can model the dynamics of convective plasma on the sun are of great interest. An attempt in this scope have been performed on a spherical surface that rotates like the sun [1]. The authors suggested a 2D and 1D linear framework of incompressible Navier-Srokes equations and formulated the modes as solutions of eigenvalue problems. The results showed that the linear model succeeded to recover a good representation of the equatorial Rossby modes and the high latitude inertial modes, while the critical latitude inertial modes were not well represented. In order to get over the limitations of the linearity assumption, it's more convenient to account for the convective term effects in the equations. A first attempt to take into account the convective term in the Navier-Stokes equations over a curved surface has been done in [2, 3]. Given the spherical geometry of the domain, the authors used DEC for discretization provided its adequate description of integration and differentiation over topology and geometry of finite discrete cell complexes and their dual [4, 5, 6].
In the present paper we extend this study and examine the modes recovery by considering in addition to the nonlinear convective term, a source term represented by a Markovian process.

To this end, we numerically simulate a two-dimensional incompressible flow that evolves freely from a prescribed initial condition [7]. Starting from a randomly generated initial condition, we show that the proposed numerical approach produces pertinent solutions that respect the rotational framework of the domain.
The paper is organized as follows. In Section 2 we introduce the continuous incompressible Navier-Stokes equations with the considered forcing term. In section 3, we present the DEC discretization of the continuous equations. Section 4 is devoted to the numerical analysis of the forced 2D turbulent flow in a rotating sphere. Finally conclusions are drawn in Section 5.

2 Problem settings

Consider the homogeneous fluid flow with uniform density ρ, evolving on the surface of the unit sphere. The flow is three dimensional and governed by the incompressible Navier-Stokes equations

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \ , \\ &\nabla\cdot\mathbf{v} = 0 \ , \end{aligned} \tag{1}$$

where $\mathbf{v}$ refers to the flow velocity, p to the gas pressure and ν to the kinematic viscosity. In a rotating frame with angular velocity $\Omega\mathbf{e}_z$, the momentum equation in (1) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\Omega\times\mathbf{v} + \nabla\Phi_{\mathrm{ce}} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \ , \tag{2}$$

where $\mathbf{r}$ is the position vector and Φ_{ce} the centrifugal potential given by

$$\Phi_{\mathrm{ce}} \equiv -\frac{1}{2}(\Omega\times\mathbf{r})^2 .$$

This quantity can be absorbed in the pressure term leading to the introduction of the effective pressure

$$p_{\mathrm{eff}} \equiv p + \rho\Phi_{\mathrm{ce}} \ . \tag{3}$$

For a flavor on scaling, the non-dimensional form of equations (2) can be written as

$$\frac{\partial \mathbf{v}^\star}{\partial t^\star} + (\mathbf{v}^\star\cdot\nabla)\mathbf{v}^\star + \frac{1}{\mathrm{Ro}}\hat{k}\times\mathbf{v}^\star = -\nabla p^\star_{eff} + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{v}^\star \ , \tag{4}$$

$$\nabla\cdot\mathbf{v}^\star = 0 \ , \tag{5}$$

where the non-dimensional numbers Re and Ro denote, respectively, the Reynolds number and Rossby number. They are defined as follows

$$\mathrm{Re} \equiv \frac{UR}{\nu}, \qquad \mathrm{Ro} \equiv \frac{U}{2\Omega R}$$

with U the characteristic velocity assumed equal to square root of total kinetic energy, and R the characteristic length scale assumed to be the radius of the sphere. In what follows and for the sake of clarity, stars upperscripts corresponding to the non-dimensionalization are omitted from variables. On a compact smooth Riemannian surface, the two-dimensional vector calculus notation of Navier-Stokes equations in a rotating frame of reference writes as follows

$$\frac{\partial v}{\partial t} - \frac{1}{\mathrm{Re}}\left[-\Delta^{dR} v + 2\kappa v\right] + \nabla_v v + \mathrm{grad}_S p_{eff} + \frac{1}{\mathrm{Ro}}\hat{k} \times v = 0, \tag{6}$$

$$\mathrm{div}_S v = 0, \tag{7}$$

where v is the tangential surface velocity, Δ^{dR} the surface Laplace-DeRham operator, κ the surface Gaussian curvature, ∇_v the covariant directional derivative, grad_S the surface gradient and div_S the surface divergence. To take into account the effect of the vertical forces and their action on the horizontal movement of the the sun, a forcing term is added to the momentum equation [8]. The forcing vector is expressed in the vorticity momentum equation as $F_\xi(\theta, \phi, t)$ (the index ξ to note that this forcing is in the vorticity formulation), it is represented by a Markovian process as

$$F_\xi(\theta, \phi, t^{n+1}) = rF_\xi(\theta, \phi, t^n) + \sqrt{1-r^2}\hat{F}(\theta, \phi, t^{n+1}).$$

Here, r is a memory coefficient related to memory timescale and $\hat{F}$ is a vorticity source generated randomly at each time step in terms of the spherical harmonics $Y_l^m(\theta, \phi)$ as follows

$$\hat{F}(\theta, \phi, t^n) = \sum_{l=l_c-\Delta l}^{l_c+\Delta l} \sum_{\substack{m=-l \\ m\neq 0}}^{+l} \hat{F}_l^m(t^n) Y_l^m(\theta, \phi).$$

The term $\hat{F}_l^m(t^n)$ is an expansion coefficient of $\hat{F}$, usually computed by setting its amplitude and phase randomly at every time step (in order to construct a homogeneous and isotropic forcing). In this case we need that the coefficients $\hat{F}_l^m(t^n)$ be computed in taking on consideration the constraints $\hat{F}_l^m(t^n) \sim N(0, \sigma_F)$, where $N(0, \sigma_F)$ is the normal distribution. In order to include the corresponding forcing term F_u (the index u to note that this forcing term correspond to the velocity formulation) in the velocity momentum equation, we need to use the relations between vorticity and velocity. The first step is to solve the problem $\Delta F_\psi = F_\xi$, where F_ψ is the stream function forcing term. Then F_u is deduced by applying the curl operator to F_ψ. The final form of the momentum equation in our case is written

$$\frac{\partial v}{\partial t} - \frac{1}{\mathrm{Re}}\left[-\Delta^{dR} v + 2\kappa v\right] + \nabla_v v + \mathrm{grad}_S p_{eff} + f\hat{k} \times v + \nabla \times F_\psi = 0 \tag{8}$$

$$\mathrm{div}_S v = 0.$$

2.1 Exterior calculus discretization

In exterior calculus notation, the previous equations (8) are written

$$\frac{\partial u}{\partial t} - \mu * \mathrm{d} * \mathrm{d}\, u - 2\mu\kappa u + *(u \wedge * \mathrm{d}\, u) + \mathrm{d}\, p^d + \star(u \wedge * f_{dual2}) + (* \mathrm{d}\, F_\psi) = 0 \tag{9}$$

$$* \mathrm{d} * u = 0 \tag{10}$$

Here, u is the one-form velocity $u = v^\flat$, p^d is the effective dynamic pressure zero-form, and the Coriolis force $f\hat{k} \times v = *(u \wedge * f_{dual2})$, where f_{dual2} is the two-form corresponding to f. The operator d is the exterior derivative. An important property of the exterior derivative operator d is $\mathrm{d}\,\mathrm{d} = 0$. In vector calculus, this property translates to $\mathbf{div}\,\mathbf{grad} = \mathbf{0}$. In $\mathbf{R}^3$, it gathers the two relations $\mathbf{curl}\,\mathbf{grad} = \mathbf{0}$ and $\mathbf{div}\,\mathbf{curl} = \mathbf{0}$. The discrete d operator verifies relation $\mathrm{d}\,\mathrm{d} = 0$, similarly to its continuous counterpart. It guarantees by construction the compatibility

of the discretization of operators div, grad and curl. Verifying this relation at the discrete level avoids having parasitic or spurious solutions, and so DEC is known to exactly conserve mass, vorticity and kinetic energy. An important operator needed to express constitutive laws in exterior calculus is the Hodge star operator "$*$". In the discrete scale, a dual mesh is needed for the definition of the discrete Hodge star operator. A popular choice of dual mesh in DEC is the circumcentric dual. For more details, readers are invited to see [4]. Equations (9) and (10) correspond to smooth exterior calculus expressions and are still continuous.
By considering a domain discretization with a primal simplicial mesh and its corresponding dual mesh, we define the discrete forms on the mesh, the discrete forms are now the integral quantities on the mesh elements. Note that presently we consider a circumcentric dual. For an incompressible flow of a homogeneous fluid with unit density, subjected to external forcing and large scale dissipation, the discrete exterior calculus expression of the governing equations, in a rotating frame of reference are as follows

$$\begin{aligned}\left[\left(-\tfrac{1}{\Delta t}\right) I + \mu K + \tfrac{1}{2}\mu\, \mathrm{d}_0 \star_0^{-1} \left[-\,\mathrm{d}_0^T\right] \star_1 - \tfrac{1}{2} \left(W_V\right)^{n+1} \star_0^{-1} \left[-\,\mathrm{d}_0^T\right] \star_1\right] \left(U^*\right)^{n+1} \\ + \star_1^{-1}\, \mathrm{d}_1^T \left(P^d\right)^{n+1/2} = F,\end{aligned} \tag{11}$$

with

$$\begin{aligned}F = \left(-\tfrac{1}{\Delta t}\right) \left(U^*\right)^n - \mu K \left(U^*\right)^n - \tfrac{1}{2}\mu\, \mathrm{d}_0 \star_0^{-1} \left[-\,\mathrm{d}_0^T\right] \star_1 \left(U^*\right)^n \\ + \tfrac{1}{2} \left(W_V\right)^n \star_0^{-1} \left(\left[-\,\mathrm{d}_0^T\right] \star_1 \left(U^*\right)^n\right) \\ + \tfrac{1}{2} \left[\left(W_V\right)^{n+1} + \left(W_V\right)^n\right] \star_0^{-1} f_{dual2} + \mathrm{d}_0\, F_\psi^{n+1},\end{aligned} \tag{12}$$

$$[\mathrm{d}_1] \left(U^*\right)^{n+1} + [0] \left(P^d\right)^{n+1/2} = 0, \tag{13}$$

Here U^* is the vector containing mass flux primal one-form for all mesh primal edges, V is the vector containing the discrete primal velocity one-forms for all mesh primal edges, and P^d is the vector containing discrete dynamic pressure zero-forms for all mesh dual vertices. Δt is the discrete time interval. The set of nonlinear equations (11) - (13) is solved using Picard's iterative method for the mass flux (U^*) and dynamic pressure $\left(P^d\right)$ degrees of freedom. More details of discretization are available in [3].

3 Numerical results

In this section, preliminary numerical results of the two dimensional forced turbulence flow problem are presented. The rotation rate of the frame is set to $\Omega = \Omega_{\text{Carr}} = 445$ nHz, which correpsonds to a Rossby number $Ro = 0.39$. The turbulent Reynolds number in our case is $Re = 300$. The numerical computations are performed on the the unit sphere[1] composed of well centered triangular mesh, see Figure 1. To start the flow, an arbitrary initial velocity condition that models the behaviour of the sun needs to be set. Previous choices in this regard has been made. For instance, *Jagad et al.* used a horizontal wave velocity formed by spherical harmonics [2]. In [9], the authors used a stream function distribution corresponding to the background zonal flow. In the present paper we chose to use a combination of both terms. Thus, The

[1]Note that computations are performed on the nondimensional variable. These are obtained by using the the radius of the sun $R = 7e8$ m as characteristic length.

initial stream function distribution computed as the sum of the horizontal wave velocity and the background zonal flow. In spherical harmonics, the degree l characterizes the total wave number and order m characterizes the azimuthal/zonal wave number of the spherical harmonics. By choosing a wave number bounded between $l = 4$ and $l = 10$, the horizontal wave velocity is given as follows

$$\Psi(\theta, \phi) = \sum_{l=4}^{10} \sum_{m=-l}^{l} \Psi_{lm} Y_m^l(\theta, \phi), \tag{14}$$

where θ is the colatitude, ϕ is the longitude, Y_l^m is the spherical harmonic function of degree l and order m, and Ψ_{lm} are the expansion coefficients or spherical harmonic modes verifying $\Psi_{l-m} = (-1)^m \Psi_{lm}^*$, with Ψ_{lm}^* the complex conjugate of Ψ_{lm}. The values of the coefficients Ψ_{lm} are chosen as in [7].
The background flow velocity is related to the differential rotation profile through $U(\theta) \equiv R \sin\theta\,(\Omega(\theta) - \Omega)$, where Ω is the angular velocity of the rotating frame. To this end, we assume a simple differential rotation profile

$$\Omega(\theta) = \Omega_0 + \Omega_2 \cos^2\theta + \Omega_4 \cos^4\theta \ . \tag{15}$$

As a result, the stream function corresponding to the background zonal flow writes

$$\psi_{\text{background}}(\theta) = \int_0^\theta d\theta' \ R\, U(\theta') \ . \tag{16}$$

The initial mass flux 1-form for a primal mesh edge is now computed as

$$u^* = d_0(\Psi + \psi_{\text{background}})$$

and the vorticity distribution at the primal mesh nodes is computed as

$$\omega = d_*^1[-d_0^T] *_1 u^*$$

An illustration of the initial condition of the simulated flow is given in Figure 2. It shows the initial vorticity distribution in the sphere from the northern (left) and southern viewpoints (right). One can observe the existence of local concentrated zones with high/low vorticity intensities. These zones describe the local spinning motion resulting from the combination of the horizontal wave velocity and the background zonal flow. In order to inspect the flow prediction in time, we inspect the normalized kinetic energy difference given by

$$\textbf{Kinetic energy} = \frac{KE(t) - KE(t=0)}{KE(t=0)}$$

Figure 3 shows the evolution of the kinetic energy as function of the nondimensional time. It can be observed that the energy decreases and reaches a quasi-stagnation regime after $t = 2.5$. Further in time, the energy oscillates but remains bounded in a narrow interval of values meaning that the simulation converged towards a certain stabilization regime. We refer to the solution at this regime as the stationary solution. Figure 4 displays the vorticity of the solution at $t = 20$ from different viewpoints, east, west, south, north, front and back. A visual inspection of the results certifies the pertinence of the proposed numerical framework for the prediction of the 2D Turbulence flows on a sphere. It can be seen that the flow respects a spinning pattern following the direction of the rotation of the sphere. More specifically, in the north side of the sphere, one can observe that the spinning intensities are the lowest. Then, by moving up along the latitudinal direction, the flow propagates in a spinning way and the velocities become higher until reaching a concentrated region at southern region with the maximum velocities.

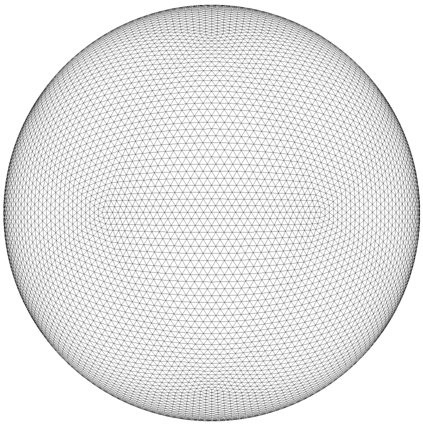

Figure 1: Well centered mesh of the unit sphere as a simplicial mesh.

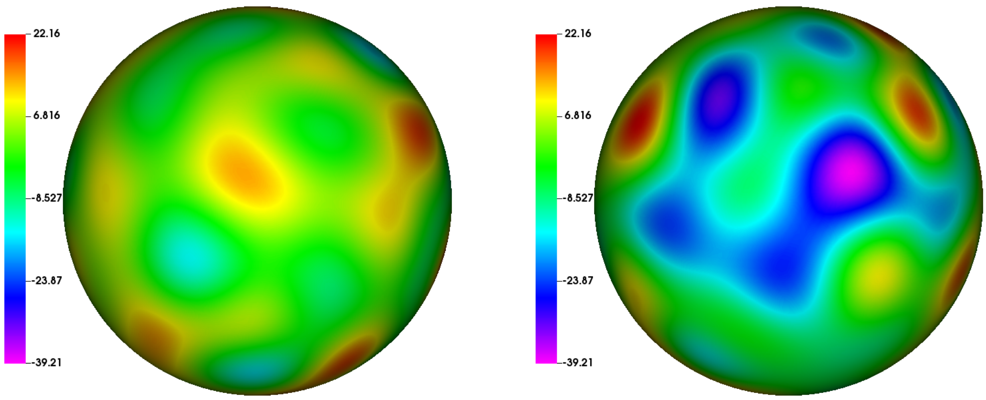

Figure 2: Initial vorticity field computed from the spherical harmonic modes and background zonal flow. Viewpoints are respectively from south (left) and north (right).

4 Conclusion and Future Work

In the present work, the numerical simulation of 2D turbulent Flows by DEC is motivated. As an extension to previous studies, we added a forcing term to the momentum equation to take into account the effect of the vertical forces and their impact on the horizontal movement of the the Sun. The obtained results are promising and open the perspective to more challenging investigations, such as considering finer mesh and computing high resolution vorticity power spectra to identify the inertial and Rossby waves in the flow. These modes will be subject to study and comparison with those derived from observations and existing established linear models.

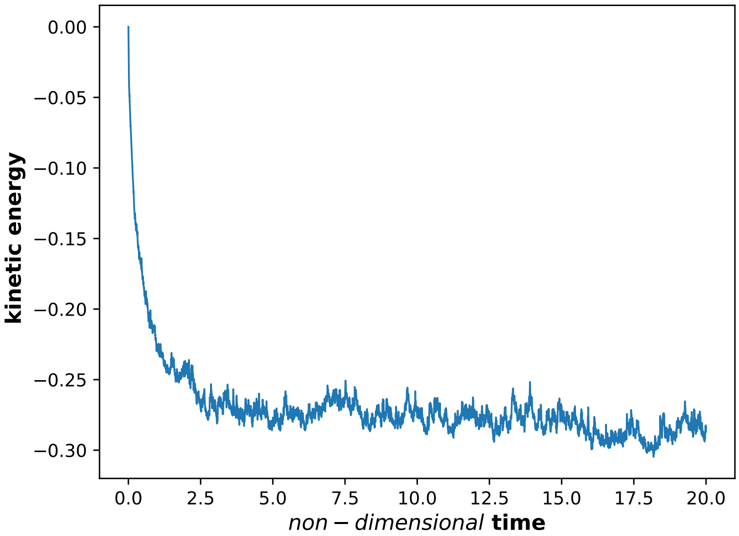

Figure 3: The stationary state kinetic energy for the 2D forced turbulence flow.

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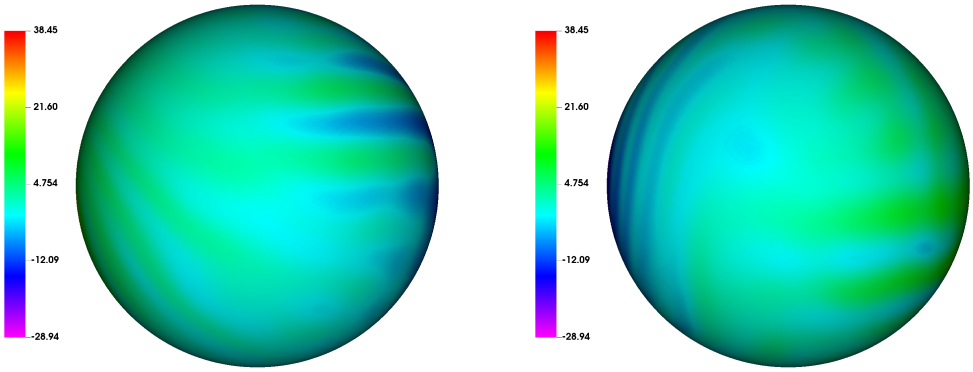
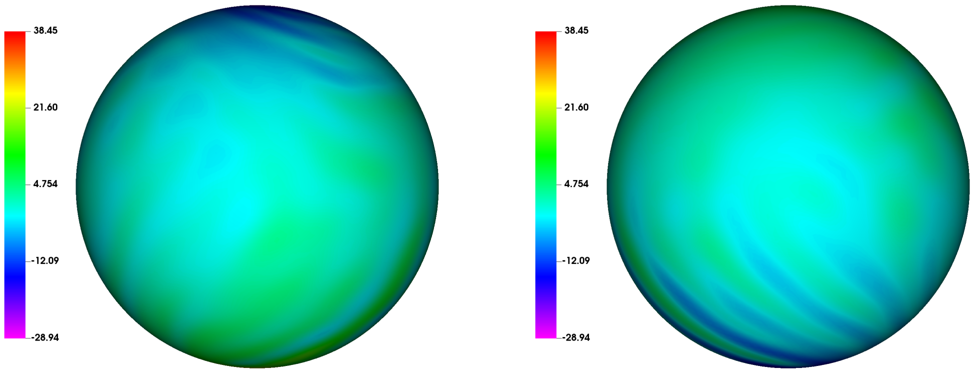
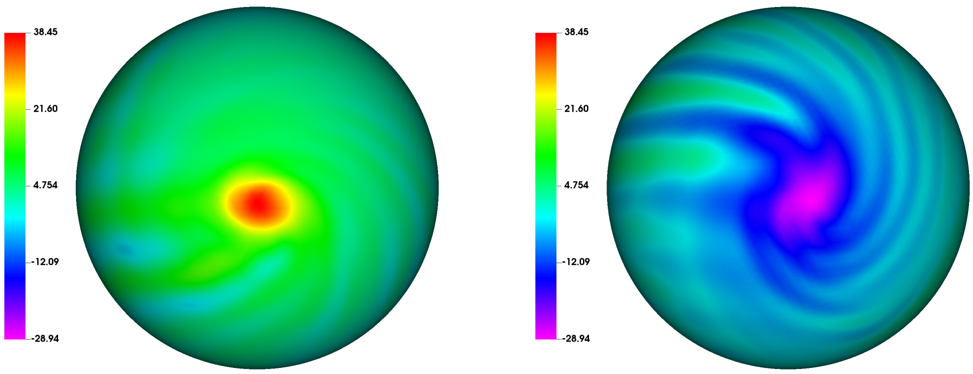


Front/Back viewpoints



West/East viewpoints



South/North view points

Figure 4: Evolution of the vorticity field distribution in the stationary regime.