Recent Development of Entropy Split Methods for Gas Dynamics and MHD

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Abstract Sjögreen and Yee [7, 8] proved that the high order entropy split methods based on Harten's entropy function [2] are entropy stable for central differencing for the nonlinear thermally-perfect gas dynamics Euler equations. Harten's entropy function also can be part of the Tadmor-type entropy conserving method [5, 7]. Here the same Harten entropy function is further developed the ideal MHD (magnetohydrodynamics) governing equation set. The present study concentrates on the comparative study the recent development of entropy split methods of Yee et al., Sjogreen & Yee entropy split methods [11, 7, 12, 8, 9] with the Tadmor-type of discrete entropy conserving [10], momentum conserving [1], kinetic energy preserving [3], and a combination of these physical-preserving methods. Some of these high order methods belong to the class of skew-symmetric splitting of the inviscid flux derivatives [9]. All of these methods not only preserve certain physical properties of the chosen governing equations but are also known to either improve numerical stability, and/or minimize aliasing errors in long time integration for turbulent flow simulations without the aid of added numerical dissipation for selected flow types. The nomenclature for spatial discretizations for comparative studies are:

- ECLOG - The Tadmore-type entropy conservative method (EC) with logarithmic entropy $E_L = -\rho \log(p\rho^{-\gamma})$ but without the kinetic energy preserving modification [5]
- ECLOGKP - Tadmor-type method for E_L with Ranocha's kinetic energy preserving modification [4]
- ECBKP - The Tadmor type entropy conservative (EC) method for Harten's entropy $E_H = -\frac{\gamma+\alpha}{\gamma-1}\rho(p\rho^{-\gamma})^{\frac{1}{\alpha+\gamma}}$. It turned out that this method in its base form also satisfies Ranocha's kinetic energy preservation condition; so there is only one variant for this method [5, 4].
- ES - The entropy split method for E_H on e.g., $\mathbf{F}_x = \frac{\beta}{\beta+1}\mathbf{F}_x + \frac{1}{\beta+1}\mathbf{F}_W W_x$, where W is the entropy variable vector, F_x is the Euler flux derivative in the x-direction, with β a positive splitting parameter. For shock-free turbulence, $\beta = 1$ to $\beta = 2.5$ are used The central scheme is applied to the entropy splitting form of the convection flux derivatives– a skew-symmetric splitting form as well [11, 7].
- ESSW: Entropy split with Ducros et al. splitting but switch to regular central near discontinuities [8]
- DS - The Ducros et al. splitting of the convection flux derivatives [1].
- ESDS - The entropy conservative split method with Ducros et al. splitting (DS) used on the conservative portion of the entropy split flux derivatives. This is a heuristically split method as it is a partial Ducros et al. split.
- DSKP - The Ducros et al. splitting of the convection flux derivatives with kinetic energy preservation
- KGS - The Kennedy-Gruber-Pirozzoli splitting of the convection flux derivative. This is the form that is kinetic energy preserving and that can be written in conservative form using numerical fluxes [3].

Representative test cases for shock-free turbulence and turbulence with shocks were conducted for the gas dynamics and MHD test cases with smooth flows and flows with discontinuities. With the current studies and from our previous studies [7, 12, 8], EC and ECLOG, ECLOGKP are the most CPU intensive methods among the nine methods. They are approximately twice the CPU per time step than the ES, ESDS and ESSW methods and yet exhibit similar resolutions. DS is the least CPU intensive. The test cases set up are the same as in [11, 7, 8] with the classical fourth-order Runge-Kutta time discretization. For illustration purposes, the standard gas dynamics shock-free turbulence test case – 3D Taylor-Green vortex – is chosen using the end time 20 instead of the standard end time 10 to observe the solution behavior twice as long. Figure 1 shows the closeup comparison of kinetic energy, entropy E_H, maximum-norm of error, and entropy

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E_L vs. time for the eight methods. The kinetic energy and entropy results show the quantity with its value at time zero subtracted, e.g., the kinetic energy ($E_{kin(t)}$) shown is $E_{kin}(t) - E_{kin}(0)$. The ES method starts to lose some energy at a later time. Otherwise the stable results are similar. ECLOGKP and KGS are indistinguishable, and fall on top of each other. Overall, ECLOGKP, ECLOG, ECBKP, KGS, and DS are very similar. However, differences might be larger for other flow problems. For the 2D MHD Alfvén wave test

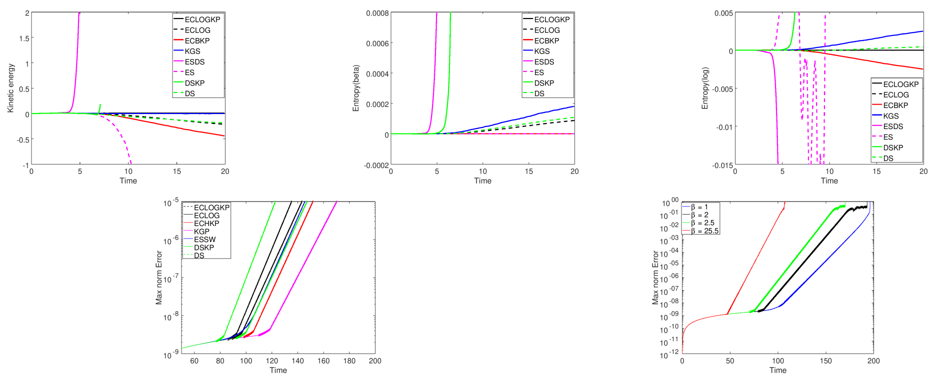

Fig. 1 Inviscid 3D Taylor-Green Vortex with 64^3 grid points (top) and MHD Alfvén wave (bottom): Top: Comparison of kinetic energy, entropy E_H, and entropy E_L vs. time for the 8 methods for the Taylor-Green Vortex. Bottom: maximum norm error vs. time for the MHD Alfvén wave.

case, the bottom of Figure 1 shows the time evolution of the maximum norm error over the computational domain for seven methods. The error is maintained at a very low level up to time around 130 for all the methods. Furthermore, at times before 130, the different high order numerical schemes produce errors that are indistinguishable. However EC and ECLOG, ECLOGKP are the most CPU intensive methods among the nine methods with approximately twice the CPU per time step than the ES, ESDS and ESSW methods and yet exhibit similar resolutions.

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