Synthesizing Turbulent Channel Flow

A Proposed Form of Solution to Steady Turbulence in Channels

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Abstract: There has long been a need for a fully theoretical basis for describing turbulent flows. We propose a formal representation of the random velocities using a few ordinary smooth non-random functions together with an ordinary Stochastic Integral. We call this the ***Gaussian Transform*** or ***GXF***. For example, for **3D** Channel Flow, we need only two **CDF**'s (Cumulative Distribution Functions) and four ordinary correlation functions. The simpler case of **2D** Channel Flow requires only one **CDF** and one correlation function. Decaying Isotropic Turbulence (**HIT**) requires only one **CDF** and two correlations functions. We will develop these functions for **2D** Channel Flow; we will describe how to compute them numerically from **DNS** results; we will demonstrate the synthesis of multiple realizations of the flow; and we will apply the **NSE** to this representation to come to a full solution to **2D** Channel Flow.

Keywords: Theoretical Turbulence, Multidimensional Stochastic Integrals, Ritz-Galerkin approximation, Metalog **CDF** approximations, Stationary Ergodic Stochastic Processes

1 Introduction

This paper is divided into two main parts and a conclusion section:

First a theoretical development of the **GXF** method for **2D** Channel Flows with adequate (perhaps heuristic) rigor. We will use the **CDF**; the corresponding quantile functions; and then the corresponding normal random variable. From this we can compute the kernel of the ***Wiener Stochastic Convolution Integral***.

Second, we determine numerically the two functions using **DNS** results. The **CDF** is obtained from recorded velocities **u[i,x,y,t]** (Python notation) by simple algebra or perhaps metalog mechanisms, thence the numerical quantile function, thence the numerical Auto-Correlation, thence the kernel of the Wiener Integral.

In conclusion, we discuss analytic representation of the **CDF** and auto-correlation functions. These can be plugged into the **NSE** and optimized using (say) Ritz-Galerkin methods for a theoretical solution to this **2D** Channel Flow Problem. There are many more flow situations which are amenable to this analysis, including decaying flows, **MFD** flows, compressible flows, and many other internal and external flows.

2 Developing the Gaussian Transform

Consider **2D** Channel Flow with the usual assumptions: Newtonian Fluid, no-slip walls, incompressible, obeying the Navier Stokes equations, etc. The flow is stable, stationary on **x** (stream-wise) and **t**. Reynold's averaging applies, i.e., the flow is "***ergodic***" on **x** and **t**.

To illustrate the process: place a velocity probe at some point $\{\mathbf{x_0},\mathbf{y_0}\}$ and record the velocity **u(t)** against time. This random $\mathbf{u(t,\alpha)}$ (including a sampling parameter α) has a smooth **CDF** so that:

$$
\begin{aligned}
&\Pr\left[u(t,\alpha)\le\lambda\right]=CDF(\lambda) && \text{Smooth Strictly Increasing}\\
&q(t,\alpha)=CDF\left(u(t,\alpha)\right) && \text{Derived Quantile RV, has linear cdf}\\
&\phi(t,\alpha)=\Psi^{-1}\left(q(t,\alpha)\right) && \text{Derived Normal RV, has Normal cdf}\\
&\phi(t,\alpha)=\int\Phi(t-\tau)\,dr(\tau,\alpha) && \text{Wiener Stochastic Convolution Integral}
\end{aligned}
\qquad (1)
$$

Here $\mathbf{\Psi^{-1}(\lambda)}$ is the inverse Normal distribution. The $\mathbf{\Phi(\tau)}$ kernel is determined uniquely from the auto-correlation of $\mathbf{u[t,\alpha]}$. Thus, given a **CDF** and an auto-correlation function, any stationary ergodic random variable can be synthesized by reversing the process of equation (1) thus:.

$$
\begin{aligned}
&\phi(t,\alpha) = \int \Phi(t-\tau)\,dr(\tau,\alpha) && \text{Wiener Stochastic Convolution Integral} \\
&q(t,\alpha) = \Psi\left(\phi(t,\alpha)\right) && \text{Generated Normal-Quantile RV, has linear cdf} \\
&u(t,\alpha) = CDF^{-1}\left(q(t,\alpha)\right) && \text{Generated Physical RV, has physical cdf}
\end{aligned} \tag{2}
$$

By a straightforward extension, **2D** Channel Flow can be synthesized as follows

$$
\begin{aligned}
&\phi(x,y,t,\alpha) = \int \Phi(x-\varepsilon, t-\tau, y)\,dr(\varepsilon,\tau,\alpha) && \text{2D Wiener Integral, Normal RV} \\
&q(x,y,t,\alpha) = \Psi\left(\phi(x,y,t,\alpha)\right) && \text{Generated Normal-Quantile RV} \\
&s(x,y,t,\alpha) = CDF^{-1}\left(q(x,y,t,\alpha), y\right) && \text{Generated Physical Function RV} \\
&u^{x}(x,y,t,\alpha) = -\partial_{y}\, s(x,y,t,\alpha) && \text{u is "curl" of s} \\
&u^{y}(x,y,t,\alpha) = +\partial_{x}\, s(x,y,t,\alpha) &&
\end{aligned} \tag{3}
$$

So, we claim, **2D** channel turbulence is completely described by these two functions: **CDF(λ,y)** and **Φ(x,t,y)**

The extension to **3D** Channel Flow turbulence is straightforward but complex in detail. The **3D** case requires two **CDF(λ,y)** and four **Φ(x,t,z,y).**

3 Using the Gaussian Transform Numerically and Analytically

A robust **DNS** run for **2D** Channel Flow turbulence will provide the data for a numerical description of both the **CDF(λ,y)** and **Φ(x,t,y).** There is much work to be done to compute these two functions over all **y**-values. And then for several **Re** values. But the results will allow synthesis of any realization of the flow.

Also, the **CDF**'s can be approximated by a variety of techniques – most promising is ***Metalog*** (see Wikipedia) with a few parameters. And the auto-correlation functions can be approximated by a sum of Hermite Functions or other rapidly decreasing functions – hopefully with a few parameters. This will give good insight into the structure of both functions against **y**-values and **Re**-values.

A third prospect is this: construct the approximation just above (or better); insert into the **NSE** and minimize the mean-square of the residual – a ***Ritz-Galerkin*** model. Hopefully a few parameters will suffice, and the results will be close to the **DNS** results.

Much more to be said. Many results forthcoming.

4 Conclusion and Future Work

We have shown that there is an ***Analytic Representation of Stationary Turbulent Channel Flows***. We have outlined the construction of this mechanism using **2D** Channel Flow as an exemplar. The methods apply to a variety of internal and external flows.

We will continue the investigation using **2D** Channel Flows to determine dependence on **Re** and to develop better approximations to **CDF(λ,y)** and **Φ(x,t,y).** Ultimately – time and energy permitting – we will apply the **NSE** to this formulation most likely using a Ritz-Galerkin best fit technique.

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