Property-preserving limiters for discontinuous Galerkin discretizations of hyperbolic problems

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Abstract: We consider discontinuous Galerkin (DG) discretizations of hyperbolic conservation laws. The piecewise-constant ($\mathbb{P}_0$) version corresponds to a low-order finite volume method, which is entropy stable and locally bound preserving. A piecewise-linear ($\mathbb{P}_1$) or higher order DG approximation preserves these nonlinear stability properties if it is equipped with properly designed flux and/or slope limiters. We formulate the corresponding inequality constraints and show how they can be enforced in practice. In particular, we introduce flux limiters that ensure the validity of discrete maximum principles and cell entropy inequalities for cell averages. The partial derivatives of the DG-$\mathbb{P}_1$ approximation are constrained using a slope limiter to ensure preservation of entropy stability and boundedness of partial derivatives in terms of low-order reconstructions. We also discuss the possibility of slope limiting subject to flux constraints or local maximum principles for pointwise solution values. The combination of direct flux limiting with slope limiting under derivative constraints produces the best results in our numerical experiments.

Keywords: Discontinuous Galerkin Methods, Maximum Principles, Positivity Preservation, Entropy Stability, Flux Correction, Slope Limiting.

Outline

In contrast to finite volume methods, DG-$\mathbb{P}_1$ schemes evolve not only cell averages but also partial derivatives of polynomial approximations $u_{ih} \in \mathbb{P}_1(K_i)$ on mesh cells K_i. The semi-discrete local problem associated with a single cell can be written as (see [1] for details)

$$\int_{K_i} w_{ih} \frac{\partial u_{ih}}{\partial t}\,\mathrm{d}\mathbf{x} + \sum_{j\in\mathcal{N}_i} \int_{S_{ij}} w_{ih} H_{ij}^{\mathbb{P}_1}\,\mathrm{d}s - \int_{K_i} \nabla w_{ih}\cdot \mathbf{f}(u_{ih})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall w_{ih}\in\mathbb{P}_1(K_i),$$

where $\mathbf{f}(u)$ is the flux function and $H_{ij}^{\mathbb{P}_1} = H_{\mathrm{LLF}}(u_{ih}, u_{jh}; \mathbf{n}_{ij})$ is the local Lax–Friedrichs (LLF) flux across an edge/face $S_{ij} \in \partial K$ with the unit outward normal $\mathbf{n}_{ij}$. The set $\mathcal{N}_i$ contains the indices of all common edge/face neighbors. The average value U_{i0} of u_{ih} in K_i is known to satisfy all relevant constraints if $H_{ij}^{\mathbb{P}_1}$ is replaced by $H_{ij}^{\mathbb{P}_0} = H_{\mathrm{LLF}}(U_{i0}, U_{j0}; \mathbf{n}_{ij})$. Hence, these constraints can always be enforced by using a property-preserving *flux limiter* to construct

$$H_{ij}^* = \alpha_{ij} H_{\rm LLF}(u_{ih}^*, u_{jh}^*; \mathbf{n}_{ij}) + (1-\alpha_{ij}) H_{ij}^{\mathbb{P}_0}, \qquad \alpha_{ij} \in [0,1]$$

and/or a *slope limiter* to limit the partial derivatives $U_{ik} := \frac{\partial u_{ih}}{\partial x_k},\ k=1,\ldots,d$ as follows:

$$U_{ik}^* = \beta_{ik} U_{ik}, \qquad \beta_{ik} \in [0,1].$$

Note that H_{ij}^* is generally defined using the traces of slope-limited approximations u_{ih}^* and u_{jh}^*.

Adding an entropy correction term $P_i(w_{ih}, u_{ih}^*)$, we consider the modified DG-$\mathbb{P}_1$ scheme

$$\int_{K_i} w_{ih} \frac{\partial u_{ih}}{\partial t}\,\mathrm{d}\mathbf{x} + \sum_{j\in\mathcal{N}_i} \int_{S_{ij}} w_{ih} H_{ij}^*\,\mathrm{d}s - \int_{K_i} \nabla w_{ih} \cdot \mathbf{f}(u_{ih}^*)\,\mathrm{d}\mathbf{x} + P_i(w_{ih}, u_{ih}^*) = 0.$$

We prove the validity of local maximum principles and semi-discrete entropy inequalities for U_{i0} under sufficient conditions of the form $F_{ij}^{\min} \le F_{ij}^* \le F_{ij}^{\max}$, where $F_{ij}^{\min}$ is a nonpositive lower bound and $F_{ij}^{\max}$ is a nonnegative upper bound for the average *antidiffusive flux*

$$F_{ij}^* = \frac{1}{|S_{ij}|} \int_{S_{ij}} (H_{ij}^* - H_{ij}^{\mathbb{P}_0})\,\mathrm{d}s.$$

Since $F_{ij}^* = 0$ for $\alpha_{ij} = 0$ and/or $\beta_{i1} = \ldots = \beta_{id} = 0$, the flux constraints can always be satisfied by tuning the correction factors α_{ij} and β_{ik}. We review algorithms designed for this purpose and introduce new ones. The limiting techniques under investigation include flux limiters developed in [2, 3] in the context of continuous finite element approximations, the vertex-based Barth-Jespersen slope limiter for DG methods, as well as new slope limiters based on flux constraints and constraints for directional derivatives. The derivative-based approach makes it possible to preserve directional monotonicity in applications to problems that require different treatment of different space directions. At the flux limiting stage, the anisotropy of the problem at hand can be taken into account by using a customized definition of local bounds for inequality constraints. At the slope limiting stage, we adjust the magnitude of individual directional derivatives using low-order reconstructions from cell averages to define the bounds. In this way, we avoid unnecessary limiting of well-resolved derivatives at smooth peaks and in internal/boundary layers.

The new limiting tools can be readily extended to hyperbolic systems following [2, 3]. In the context of *hp*-adaptivity, optimal accuracy can be achieved by using flux/slope limiting in $\mathbb{P}_1$ subcells of macroelements marked as 'troubled' by a smoothness indicator.

References

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