Analysis of Edge-Based Method on Tetrahedra for Diffusion

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In this paper, we will establish consistency and formal second-order accuracy of a novel edge-based viscous (EBV) method on two-dimensional and three-dimensional simplicial grids. The EBV method has recently been introduced as an efficient alternative to a second-order accurate cell-based viscous (CBV) method currently used in NASA's FUN3D code [1], the latter is equivalent to the Galerkin linear finite-element method [2] on simplicial grids. The EBV method demonstrates significant speedup in practical aerodynamic simulations with no significant changes in the solutions but currently lacks a theoretical foundation for its accuracy. This paper reports on analytical and computational studies to build confidence in EBV solutions.

The CBV method computes diffusion fluxes in a loop over cells. At each cell, the solution gradients are computed using the solution at cell vertices. This approach results in a compact discretization stencil that is suitable for massively parallel computations and conveniently supports the exact linearization of the residual. The main disadvantage of the cell-based approach is a relatively high computational cost of the cell loop. A new EBV method has been recently derived [3] following the approach proposed by Barth [4]. In this EBV method, the viscous residual is more efficiently computed in an edge loop, and its compact stencil involves only edge neighbors. The EBV method requires a modest memory increase to store a few precomputed coefficients per edge.

The EBV method has been applied to the viscous kernel of the Reynolds-averaged Navier-Stokes (RANS) equations [3] that evaluates the viscous fluxes of the meanflow equations, the diffusion term of the turbulence-model equation, and the corresponding Jacobian terms. The time spent on the viscous-kernel computations has been reduced by more than a factor of three, and the fraction of viscous-kernel computations has been reduced from over 33% to 13% of a typical nonlinear iteration. A recent further optimization improves the EBV speedup of the viscous-kernel computations to a factor of six and reduces the fraction of viscous-kernel computations to less than 2.5%. An extension of the EBV method to mixed-element grids and chemically reacting flows is reported in Ref. [5]. Figure 1 compares EBV and CBV iterations that are performed by the baseline solver of FUN3D on a three-dimensional tetrahedral grid with about 4.6 million cells and 800 thousand grid points. The benchmark is an energetic, high-altitude, chemically reacting, hypersonic flow around a model crew-exploration vehicle with two temperature models and eleven air species. The EBV and CBV solutions show a consistently similar residual convergence per iteration, while the wall time required for the EBV solution is at least 20% less than the wall time required for the CBV solution.

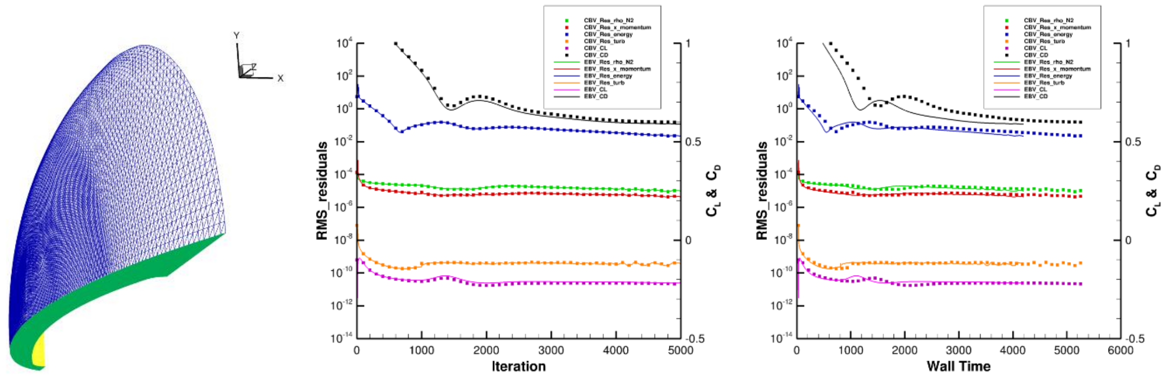


Figure 1. Baseline FUN3D iterations for benchmark hypersonic simulation.

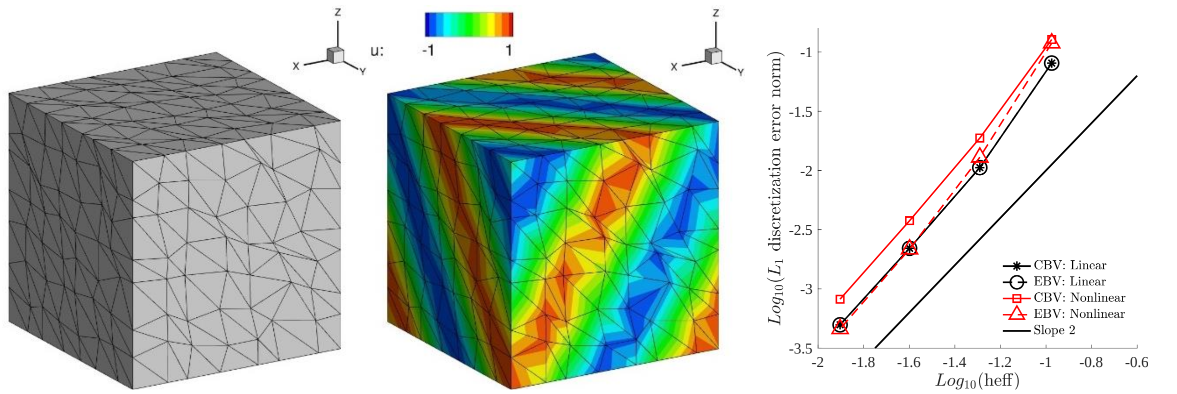
Unlike the CBV method, the EBV method is cast as summation of edge-based solution differences and does not have a clear association with flux conservation. The viscosity coefficients at the edges are computed by averaging viscosity at the edge endpoints. The authors are not aware of any theoretical foundation that can readily and rigorously support the consistency and accuracy of the EBV method for non-constant viscosity. The verifiable similarity of the EBV and CBV solutions and significant efficiency improvements offered by the EBV method warrant efforts to establish such a foundation.

References [3, 5] focus on verification of the EBV implementation and efficiency assessment for RANS solutions. This paper focuses on establishing consistency and formal second-order accuracy of EBV solutions. The full paper will present the EBV discretization for a scalar diffusion equation in detail and compare its complexity to the complexity of the CBV discretization. Analytical and computational studies will be reported to compare accuracy and grid convergence of the CBV and EBV methods on simplicial grids in application to linear and nonlinear scalar diffusion equations and to the laminar Navier-Stokes equations.

The truncation error analysis based on Taylor expansions will be conducted for the CBV and EBV methods on regular isotropic and anisotropic grids. A preliminary study shows that both methods provide the leading second-order truncation-error terms for the diffusion equation:

$$\mathrm{div}(\mu \boldsymbol{\nabla} u) = g, \tag{1}$$

where u is a differentiable function, μ is a viscosity, and g is a force function. A method of manufactured solutions has been applied to establish second-order convergence of the EBV and CBV discretization errors on families of irregular grids. See Figure 2. The linear Eq. 1 corresponds to $\mu = 1$, and the nonlinear Eq. 1 corresponds to $\mu = 1 + u^2$. Second-order convergence of discretization errors is evident. The analysis for the Navier-Stokes equations will be reported in the full paper.



Coarse grid | Manufactured solution | Discretization error convergence

Figure 2. Irregular isotropic tetrahedral grids.

References

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