Collocation Counterparts of Galerkin Methods for Ordinary Differential Equations

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Abstract (to be submitted to ICCFD 11)

Numerical methods for ordinary differential equations, also known as time stepping methods, play an essential role in engineering and scientific computing. For Computational Fluid Dynamics (CFD), as mentioned in the NASA Vision 2030 report and agreed by most experts, time stepping is one of the remaining bottlenecks for the accurate numerical simulations of turbulent flows. Among time stepping methods, two important classes are Galerkin and collocation.

The following proposition holds for the three most popular Galerkin time stepping methods: each is equivalent to a collocation counterpart up to an error of order higher than the order of the method. This observation provides intuition as well as simplifies the derivation of the resulting implicit Runge-Kutta method for a Galerkin scheme. To put it differently, the proposition establishes the relation between projection and interpolation, the key concepts of respectively the Galerkin and collocation approaches.

The general scalar ordinary differential equation (ODE) is given by

$$u'(t) = f\big(t, u(t)\big) \tag{1a}$$

with initial condition

$$u(t_0) = u_0. \tag{1b}$$

Here, the focus is on one-step methods where, with time step of size Δt, the data u^n at time $t^n = n\Delta t$ is known and the solution u^{n+1} at t^{n+1} is to be calculated.

In addition, for ease of comparison, we also focus on methods resulting in implicit Runge-Kutta schemes of s stages. In the case of collocation, these stages correspond to the s collocation points. In the case of the Galerkin methods, they together with the data u^n correspond to the $s+1$ coefficients of a polynomial solution of degree s. The exception, however, is the case of discontinuous Galerkin, where the s stages correspond to a solution polynomial of degree $s-1$ (since the data u^n is enforced only weakly).

By rescaling, we may assume $t^n = 0$ and $t^{n+1} = 1$. The data at $t^n = 0$ is denoted by u^0, and the solution polynomial $u_S(t)$ on $[0, 1]$ as well as $u_S(1) = u^1$ are to be calculated.

Concerning the collocation methods, all collocation points discussed are quadrature points on $[0, 1]$.

The first Galerkin method seeks a solution polynomial $u_G(t)$ of degree s with $u_G(0) = u^0$. The remaining s conditions to determine u_G are given by employing the test space $\boldsymbol{V}_0$ of polynomials of degree s that vanish at $t = 0$ (thus, $\boldsymbol{V}_0$ has dimension s). Concerning the corresponding collocation method, denote the $s+1$ left Radau points by $c_0, c_1, \dots, c_s$ where $c_0 = 0$.

Proposition 1. The Galerkin method with solution polynomial of degree s and test space $\boldsymbol{V}_0$ is equivalent to the collocation method with the s nonzero members of the $s+1$ left Radau points (namely $c_1, \dots, c_s$) as collocation points.

Next, the continuous Galerkin (CG) method seeks a polynomial solution of degree s denoted by u_{CG} with $u_{CG}(0) = u^0$. The remaining s conditions to determine u_{CG} are given by employing the test space $\boldsymbol{V}$ of polynomials of degree $s-1$ ($\boldsymbol{V}$ has dimension s).

Proposition 2. The CG method is equivalent to the collocation method with the s Gauss points as collocation points.

Finally, the discontinuous Galerkin (DG) method seeks a polynomial solution of degree $s-1$ denoted by u_{DG} (usually $u_{DG}(0) \neq u^0$, and u^0 is enforced weakly). The s conditions to determine u_{DG} are given by employing, like the case of CG, the test space $\boldsymbol{V}$ of polynomials of degree $s-1$ (again $\boldsymbol{V}$ has dimension s).

Proposition 3. The DG method is equivalent to the collocation method with the s right Radau points as collocation points.

The proofs of these propositions as well as the stability and accuracy analyses of these methods will be provided in the paper.

As an example, on $[0, 1]$, solve

$$u'(t) = f(t) = 1 + 8t - 12t^2 \tag{2a}$$

with initial condition

$$u(0) = u^0 = 1. \tag{2b}$$

The exact solution is straightforward:

$$u_{\text{exact}}(t) = 1 + t + 4t^2 - 4t^3. \tag{3}$$

The first Galerkin method seeks a quadratic solution

$$u_G(t) = 1 + a_1 t + a_2 t^2 \tag{4}$$

by employing the test space $\boldsymbol{V}_0$ of quadratics that vanish at $t = 0$, i.e., $b_1 t + b_2 t^2$. (Note that $u_G(0) = 1$.) Denote $(v, w) = \int_0^1 v(t)w(t)\,dt$. Using the basis $\phi_1 = t$ and $\phi_2 = t^2$ for $\boldsymbol{V}_0$, the solution u_G is given by

$$(u_G, \phi_k) = (f, \phi_k), \quad k = 1, 2. \tag{5}$$

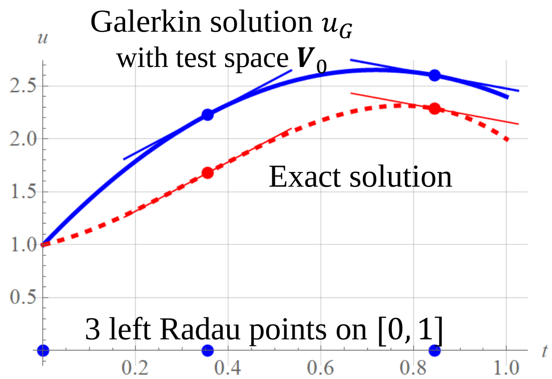
A little algebra yields

$$u_G(t) = 1 + 4.6t - 3.2t^2. \tag{6}$$

Concerning the corresponding collocation method, the three left Radau points on $[0, 1]$ are 0, $c_1 \approx 0.3551$, and $c_2 \approx 0.8449$. By a straightforward calculation with $u_G{}'(t) = 4.6 - 6.4t$ and f by (2a),

$$u_G{}'(c_1) = f(c_1) \approx 2.3277 \quad \text{and} \quad u_G{}'(c_2) = f(c_2) \approx -0.8077. \tag{7}$$

The conditions $u_G(0) = 1$, $u_G{}'(c_1) = f(c_1)$ and $u_G{}'(c_2) = f(c_2)$ define a collocation method with collocation points c_1 and c_2, which are the two nonzero members of the three left Radau points on $[0, 1]$. Thus, the Galerkin method with test space $\boldsymbol{V}_0$ yields a result identical to the collocation method with the two nonzero members of the three left Radau points as collocation points. See figure below.



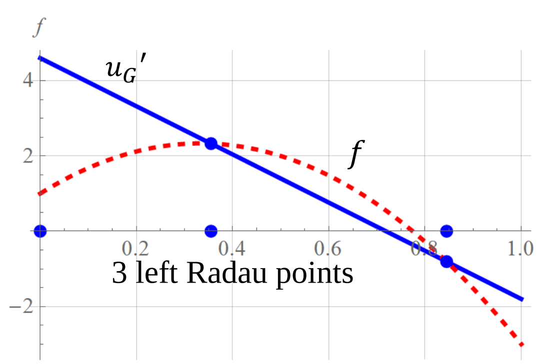


Figure. The Galerkin method with test space $\boldsymbol{V}_0$ is identical to the collocation method with the two nonzero of the three left Radau points as collocation points; left, solutions; right, corresponding derivatives.