Obtaining Accurate Functionals from High-Order Generalized Summation-by-Parts Discretizations in Curvilinear Coordinates

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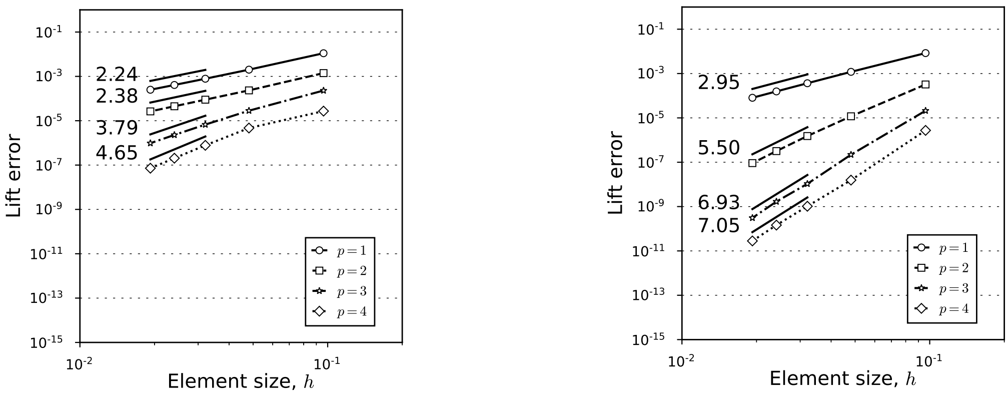
Abstract: Dual-consistent generalized summation-by-parts (GSBP) discretizations, where the discrete dual problem is a consistent discretization of the continuous dual problem, can be orders of magnitude more accurate with respect to functionals compared to dual-inconsistent discretizations. Here, several results are presented that outline how to obtain accurate functionals from dual-consistent GSBP discretizations when solving problems of increasing practical complexity. These results include showing that an optimized approach for approximating the grid metrics is important for obtaining accurate functionals with Legendre-Gauss (LG) operators, showing that the mortar-element approach outperforms the global SBP-operator approach with respect to functional accuracy in certain situations, and showing why the use of degree $p+1$ geometry representations in the presence of flow tangency boundary conditions is beneficial for both solution and functional accuracy.

Keywords: High-Order Methods, Functional Superconvergence, Generalized Summation-by-Parts, Curvilinear Coordinates.

1 Introduction

High-order large eddy simulation techniques are becoming increasingly relevant as a potential means of accurately resolving industrially pertinent problems in computational aerodynamics. Recently, GSBP operators have been introduced as a general and comprehensive means of constructing numerical schemes that can be arbitrarily high-order, conservative, and provably linearly and nonlinearly stable [1]. However, while a significant amount of attention has been devoted to the conservation and stability properties of GSBP schemes, less attention has been paid to their functional convergence properties [2]. In a previous paper [3], we showed that, in the context of linear problems, interpolation/ extrapolation operators of degree greater than or equal to $2p$ are required to preserve at least $2p$ quadrature accuracy and functional superconvergence in curvilinear coordinates when: (1) the Jacobian of the coordinate transformation is approximated by the same generalized summation-by-parts operator that is used to approximate the flux terms and (2) the degree of the GSBP operator is lower than the degree of the polynomial used to represent the geometry of interest. In this paper we outline the extension of this investigation to a suite of linear and nonlinear problems of increasing practical relevance.



Baseline metrics. Optimized metrics.

Figure 1: Lift error with LG operators using degree $p+1$ B-spline geometry representations.

2 Sample Result

As an example, consider a two-dimensional subsonic bump problem, governed by the Euler equations and discretized using GSBP operators, which uses a weighted lift on the bump surface as the functional of interest:

$$\mathcal{I}(\boldsymbol{u}) = \int_{x_1 \in [-1.5, 1.5],\, x_2 = \frac{1}{16}\mathrm{e}^{-25x_2^2}} n_{x_2}\, p(\boldsymbol{u})\, \mathrm{e}^{-8x_1^2}\, ds,$$

where $\boldsymbol{u}$ is the solution vector, p is the pressure, n_{x_2} is the x_2-component of the wall normal vector, and the Gaussian weight $\mathrm{e}^{-8x_1^2}$ is used to localize the output around the bump. Figure 1 demonstrates the importance of the method of approximating the grid metrics on functional accuracy, as the case with the baseline metrics exhibits suboptimal convergence, whereas the case with the optimized metrics exhibits significantly improved functional accuracy.

3 Conclusion and Future Work

Additional results will delineate some of the practical considerations involved in achieving accurate functionals, including showing the effect of the order of the grid, the accuracy of the wall normals, and the form of the discretization. The final results will extend [2], which demonstrated superconvergence with classical SBP operators, and [3], which showed that LG operators can exhibit suboptimal functional accuracy in certain situations.

References

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