Stable and conservative high-order methods on triangular elements using tensor-product summation-by-parts operators

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Scale-resolving simulations (i.e. direct numerical simulations and large eddy simulations) of unsteady turbulent flows largely rely on the computational efficiency achieved by spectral methods and high-order finite-difference methods when applied to smooth problems on relatively simple geometries. These methods conventionally exploit tensor-product decompositions wherein multidimensional approximation procedures are performed as sequences of one-dimensional operations, which are in part responsible for their efficiency benefits relative to inherently multidimensional finite-volume or finite-element approaches. Despite these advantages, the generation of curvilinear structured and block-structured grids, which are necessary for the application of spectral and finite-difference methods to complex geometries, remains a major bottleneck for practical flow simulations, motivating the use of multidimensional discretization techniques which are more amenable to general element types including triangles and tetrahedra. Unfortunately, multidimensional discretizations not exploiting tensor-product decompositions generally result in algorithms that impose tighter coupling between numerical degrees of freedom, which, as discussed in [1], can lead to increased computational expense relative to algorithms employing such decompositions, particularly at higher orders of accuracy.

As a critical step towards the development of efficient and robust numerical methods suitable for scale-resolving flow simulations over complex geometries, we have developed provably stable and conservative high-order methods which exploit the geometric flexibility of simplicial elements while retaining the computational benefits of a tensorial operator structure through the use of collapsed coordinate transformations, an example of which is illustrated in Figure 1. Although such transformations have been employed in the research community for several decades (see, for example, the continuous and discontinous Galerkin methods proposed in [2] and [3], respectively), the theoretical stability and conservation properties of the resulting schemes have recieved relatively little attention prior to the present work, particularly in cases involving curvilinear meshes and nonlinear or variable-coefficient problems. Restricting our attention to methods based on discontinuous solution spaces due to the advantages of their diagonal or block-diagonal mass matrices in the context of explicit temporal integration, the focus of this work is therefore on the application of recent developments in the construction and analysis of energy-stable and entropy-stable high-order methods based on the summation-by-parts (SBP) property, a discrete analogue of integration by parts, to tensor-product discretizations on triangles.

Specifically, our contributions include the development of general procedures for the construction of nodal (i.e. evolving point values) as well as modal (i.e. evolving orthogonal polynomial expansion coefficients) tensor-product operators satisfying the SBP property on triangular elements, which are used together with split formulations based on those in [4] and [5] in order to obtain conservative and energy-stable discretizations of the variable-coefficient linear advection equation in curvilinear coordinates. In the case of the nodal approach, we demonstrate that conservative and energy-stable methods of any order on straight-sided or curved triangular elements may be constructed directly from one-dimensional operators satisfying a diagonal-norm SBP property, provided that the node positions are chosen such that the singularity of the mapping is avoided. In addition to the aforementioned computational advantages of a tensorial operator structure, such nodal schemes benefit from a collocated formulation, which results in a further reduction in the cost of residual evaluation due to the numerical solution being available at the volume quadrature nodes without the need for interpolation. Moreover, the mass matrices for such schemes are diagonal, and remain diagonal even on curvilinear meshes.

For the modal approach, we provide necessary and sufficient conditions on the choices of tensor-product Jacobi-Gauss-type quadrature rules such that the SBP property is satisfied when the requisite integrals on the triangle associated with a Galerkin formulation are evaluated numerically under the mapping from the square. As discussed in [3], the use of such quadrature rules allows for an efficient tensor-product decomposition when the numerical solution is represented in terms of the Proriol-Koornwinder-Dubiner (PKD) orthogonal polynomials, which are separable under the mapping in Figure 1 in the sense that each basis function $\phi_i(\boldsymbol{\xi})$ may be decomposed as a product of polynomials in η_1 and η_2 when expressed as $\phi_i(\boldsymbol{\chi}(\boldsymbol{\eta}))$. Our approach therefore extends existing efficient modal formulations to allow for the construction of energy-stable and conservative discretizations in split form, where, contrary to the nodal approach described above, such methods contain the minimum number of local degrees of freedom for a given order of accuracy, albeit at the expense of an uncollocated formulation. We conclude with an evaluation of the accuracy and efficiency of the proposed novel tensor-product discretizations on triangles through eigensolution analysis and numerical experiments, providing comparisons between the nodal and modal approaches as well as with existing multidimensional and tensor-product schemes.

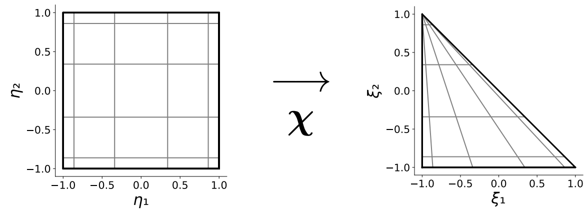

Figure 1: Mapping $\boldsymbol{\chi}(\boldsymbol{\eta}) := \left[\frac{1}{2}(1+\eta_1)(1-\eta_2) - 1, \eta_2\right]^{\mathrm{T}}$ from the square $\mathcal{Q}^2 := [-1,1]^2$ onto the triangle $\mathcal{T}^2 := \{\boldsymbol{\xi} \in [-1,1]^2 : \xi_1 + \xi_2 \leq 0\}$

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