An Implicit Algorithm for High-Order Discontinuous Galerkin Method Based on Newton/Gauss-Seidel Iterations

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Abstract: An efficient implicit algorithm was developed for high-order discontinuous Galerkin method (DGM) based on Newton/Gauss-Seidel iteration approach. The second-order to the forth order DGMs based on Taylor basis functions were employed to carry out the spatial discretization. Newton iteration scheme was used to solve the nonlinear system, and the linear system was solved with one-step Gauss-Seidel iteration. In addition, the effects of several parameters in the implicit scheme, such as the CFL number, the Newton sub-iteration steps, and the update frequency of mass-matrix, have been investigated for two-dimensional Euler equations. The numerical results demonstrate that the present implicit scheme can accelerate the convergence history evidently.

Keywords: Implicit Algorithm, Discontinuous Galerkin method, Taylor basis, Newton iteration, Gauss-Seidel iteration.

1 Introduction

Implicit time-integration schemes are highly desired to improve the efficiency, especially for high-order schemes, since they can advance the solution with significantly larger time steps comparing with the explicit methods. Many implicit schemes have been developed and applied successfully to unstructured grids to accelerate convergence to steady state [1,2]. In this paper, an efficient implicit Newton/Gauss-Seidel iteration algorithm has been developed for the high order discontinuous Galerkin method (DGM) on 2D unstructured grid. In addition, the effects of several parameters in the implicit scheme, such as the CFL number, the Newton sub-iteration steps, and the update frequency of mass-matrix, have been investigated for two-dimensional Euler equations. The numerical results demonstrate that the present implicit scheme can accelerate the convergence history evidently.

2 Newton/Gauss-Seidel iteration

The spatial discretization of DGM can be written as the following ODE form,

\[ M \frac{dU}{dt} = R(U) \] (1)

In order to solve the time-dependent problem, the resulting spatial-discretized equations must be integrated in time. In this work, the backward Euler difference was adopted as following:

\[ M \frac{\Delta U^{n+1}}{\Delta t} = R(U^{n+1}) \] (2)

in which \( \Delta U^{n+1} = U^{n+1} - U^n \) is temporal increment. An unsteady residual is defined as

\[ \text{Res}(W) = W - U^n - \Delta M^{-1} R(W) \] (3)

Equation (3) can be achieved by solving the non-linear problem \( \text{Res}(W) = 0 \) at each time step. Here we employ Newton iteration to solve (3).
where \( m \) refers to the sub-iteration index for the linear system in the Newton scheme. \( \frac{\partial \operatorname{Res}(W)}{\partial W} \) is the Jacobian matrix of unsteady residual. However, it will consume too much memory and computational cost to store and invert the Jacobian matrix of unsteady residual, if we solve (4) directly. Therefore, the second level iteration method is employed to solve (4) indirectly. In this work, a Gauss-Seidel scheme is adopted, and the Jacobian matrix is approximated by a numerical approach.

### 3 Numerical Experiments

Inviscid flow over a 2D bump from High-order CFD workshop [3] was selected to validate the convergence property firstly. The in-coming flow condition is Mach=0.5. The convergence history for the 2nd, 3rd, and 4th-order DGMs is shown in Figure 1, and compared with the third-stage TVD Runge-Kutta explicit scheme. More than one order of magnitude speedup is obtained. In Figure 2, the entropy production along the surface of the bump is plotted to indicate the solution accuracy. The second test case is the subsonic flow over NACA0012 airfoil. The in-coming flow conditions are Mach=0.5 and the angle of attack 5 degree. The mesh is shown in Fig.3a, with total of 2898 triangle cells. The convergence history of explicit and implicit 2nd–4th-order DGMs is plotted in Fig.3b. Obviously, the speedup of the implicit algorithm is very efficient.

![Convergence history](image1)

![Entropy error on the bump surface](image2)

![Computational grids and convergence history for NACA0012 airfoil case](image3)

### References

